HOW TO USE the

Model

4 VECTOR HYPERBOLIC

SLIDE RULE

by

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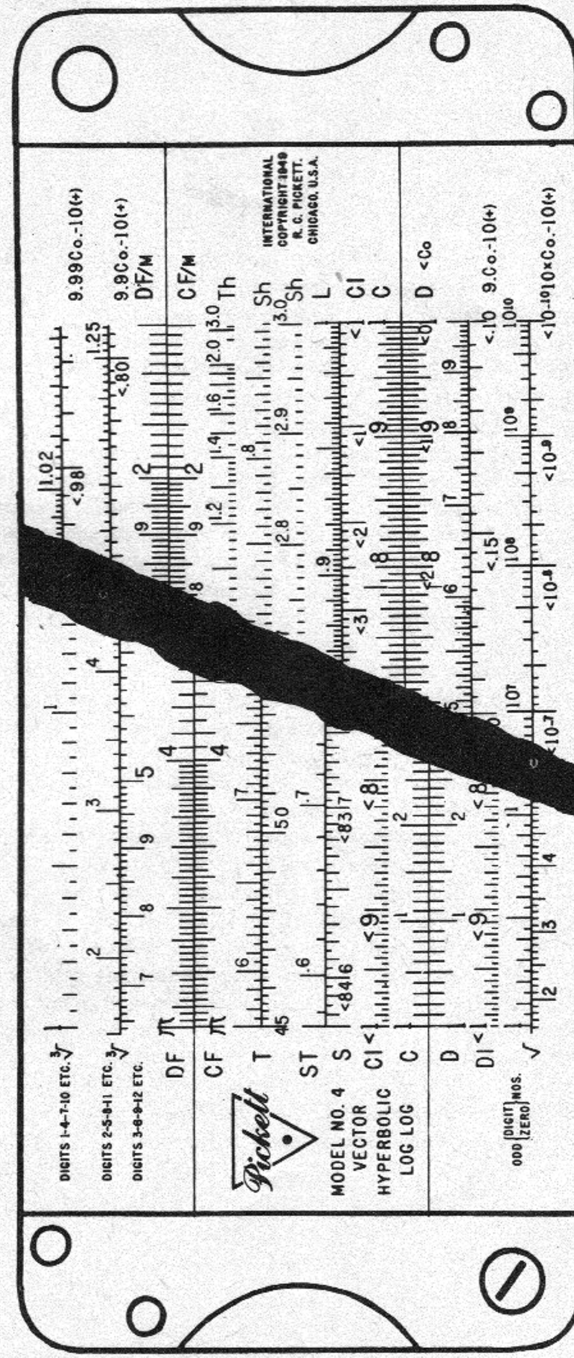

FRONT

BACK

PREFACE

In this manual for the Log Log Slide Rule, Model No. 4 (Vector Hyperbolic) the explanations are confined almost entirely to calculations involving hyperbolic functions. On the Model No. 4 rule the back of the slide contains one scale labeled Th and two scales labeled Sh which replace certain charts and legends provided on the Model No. 3 rule. Otherwise, the rules are the same. It is assumed in this manual that any reader who may need it has at hand the manual for Model No. 3. References in that manual to the use of the charts may be skipped.

Users of the hyperbolic functions will usually have need to make many vector calculations. Therefore, Part I of this manual extends the brief discussion given in the manual for Model No. 3. An alternative, and often shorter, method of calculation using the DI scale is outlined. Part 2 describes the scales used for hyperbolic functions and the elementary aspects of their use. Part 3 discusses methods of transforming sinh z, cosh z, and tanh z to the exponential form of vector representation. Part 4 treats the inverse problem of finding z such that one of these functions of z is equal to $u + jv$, where u and v are known. Part 5 contains a brief discussion of circular functions of complex arguments.

In writing this manual a choice had to be made between the following: (a) the description of methods which involve a minimum of settings but which are hard to remember and understand, and (b) descriptions which do not always lead to the least manipulation of the instrument, but which provide methods easier to understand and remember. The alternative (b) was chosen. The methods emphasized follow from formulas chosen for the analogies between them and similar formulas in more elementary mathematics. The notation has been kept uniform in order that the symbolism may help the learner keep the more important formulas in mind. The notation is, in general, that common in mathematics textbooks. This appeared to be preferable to a choice of notation which would seek to avoid all symbols used in applied mathematics with special connotations, such as the use of **Z** for characteristic impedance. Engineers and others who may use the same symbols with specialized meanings should have no difficulty in overcoming any temporary confusion which may arise on this account.

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PART 1—ELEMENTARY VECTOR METHODS

CHANGING FROM COMPONENTS TO EXPONENTIAL FORM

The following method of changing $x + jy$ to the form $\rho \underline{/\theta}$ using the DI scale is sometimes easier to use than methods based on the D scale.

Rule: **(i) To the *smaller* of the two numbers (x, y) on DI set an index of the slide. Set the indicator over the larger value on DI and read θ on the T scale. If $y < x$, then $\theta < 45°$. If $y > x$, then $\theta > 45°$ and is read from right to left (or on the left of the graduation mark).**

(ii) Move the indicator over θ on scale S (or ST), reading S on the same side of the graduation as in (i). Read ρ on DI under the hairline.

Examples:

(a) Change $2 + j\,3.46$ to exponential form. Note that $y > x$ since $3.46 > 2$, and hence $\theta > 45°$. Set right index of C over 2 on DI. Move indicator to 3.46 on DI. Read $\theta = 60°$ on T. Move indicator to 60° on S. Read $\rho = 4$ on DI. Hence $2 + j\,3.46 = 4\underline{/60°}$.

(b) Change $114 + j\,20$ to exponential form. Here $y < x$, so $\theta < 45°$. Set left index of C over 20 on DI. Move indiçator to 114 on DI. Read $\theta = 9.95°$ on T. Move hairline to 9.95° on S. Read $\rho = 116$ on DI. Hence $114 + j\,20 = 116\,\underline{/9.95°}$.

It will be observed that this rule is, in general, easy to use. In step (i) the value of tan θ for $\theta < 45°$ may be observed under the hairline on the C scale, and the value of tan θ for $\theta > 45°$ under the hairline on CI. It may be noted that the rule given in the manual for Model No. 3 obtains the result in example (b) above without having the slide project far to the right. Thus, it appears that the relative advantages of the two methods depend in part upon the problem.

LOGARITHMS OF COMPLEX NUMBERS

The logarithm of a complex number $z = x + jy$ is a complex number. Let $\log_e (x + jy) = u + jv$. Then

$x + jy = e^{u + jv} = e^u \cdot e^{jv} = e^u (\cos v + j \sin v) = e^u \cos v + j\, e^u \sin v$.

Equating the real and then the imaginary parts gives two equations

$$x = e^u \cos v$$
$$y = e^u \sin v$$

which may be solved for u and v. By division, $\tan v = y/x$, and hence $v = \arctan\,(y/x)$. Squaring and adding, $x^2 + y^2 = e^{2u}$, and hence $u = \log_e \sqrt{x^2 + y^2}$. Then

$$\log_e (x + jy) = \log_e \sqrt{x^2 + y^2} + j \arctan\,(y/x) = \log_e \rho + j\theta.$$

Rule: **To find $\log_e (x + jy)$, first convert $x + jy$ to polar form $\rho\underline{/\theta}$. Find $\log_e \rho$ and write the results in the form $\log_e \rho + j\theta$.**

Example: Find $\log_e (2.6 + j\,3.4)$. To convert $2.6 + j\,3.4$ to polar form, set right C-index over 2.6 on DI. Move indicator to 3.4 on DI. Read $\theta = 52.6°$ on T under hairline. Move indicator to 52.6° on S (reading right to left), and find $\rho = 4.28$ on DI. Set indicator on 4.28 of the $1/N_3$ scale. Read 1.454 on the $DF_{/M}$ scale. Then $\log_e (2.6 + j\,3.4) = 1.454 + j\,52.6°$, or $1.454 + j\,0.92$, when θ is in radians. This complex number may then be expressed in exponential form if desired.

PART 2. HYPERBOLIC FUNCTIONS OF REAL VARIABLES

The most important hyperbolic functions may be defined as follows. Let x be any real number and e the base of Napierian logarithms. Then:

$$\frac{e^x - e^{-x}}{2} = \sinh x \text{ ("the hyperbolic sine of } x\text{")};$$

$$\frac{e^x + e^{-x}}{2} = \cosh x \text{ ("the hyperbolic cosine of } x\text{")};$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x \text{ ("the hyperbolic tangent of } x\text{")}.$$

These functions are found useful in the application of mathematics to varied types of problems, and in particular, to problems in electrical engineering. Computations involving these functions are readily performed on a slide rule which has special scales for this purpose.

EVALUATING THE FUNCTIONS

The Hyperbolic Sine

The scales marked Sh on the slide represent values of x ranging from $x = 0.10$ to $x = 3.0$, approximately. The two scales may be viewed as one continuous scale which has been cut in half with the right hand portion placed below the left portion.

Rule: **When the indicator is set over x on an Sh scale, the corresponding value of sinh x is on the C scale under the hairline, and conversely. If the C and D scales coincide, sinh x may also be read on the D scale. If x is found on the *upper* Sh scale, the decimal point is at the left of the number as read on the C scale. In other words, $0.1 \leq \sinh x \leq 1$. If x is found on the *lower* Sh scale, the decimal point is at the right of the first digit read on the C scale. In other words, $1.0 \leq \sinh x \leq 10.0$.**

EXAMPLES:

(a) Find sinh 0.116. Set hairline over 0.116 on the upper Sh scale. Read 0.1163 on the C scale (or D scale when the indices coincide).

Verify that: sinh 0.274 = 0.277; sinh 0.543 = 0.570; sinh 0.951 = 1.100; sinh 1.425 = 1.960; sinh 2.84 = 8.53

(b) Find x if sinh x = 0.425. Set hairline of indicator over 0.425 on the C scale, read 0.413 on the upper Sh scale.

Verify that if sinh x = 6.38, then x = 2.552.

The Hyperbolic Tangent.

The scale marked Th on the slide represents values of x ranging from $x = 1.0$ to $x = 3.0$.

Rule: **When the indicator is set over x on the Th scale, the corresponding value of tanh x is on the C scale under the hairline. The decimal point is at the left of the number as read on the C scale. In other words, the approximate limits are $0.1 \leq \tanh x \leq 1.0$. For values of x greater than 3, tanh x = 1.000 to a close approximation; the error is less than one-half of 1% and decreases rapidly.**

EXAMPLES:

(a) Find tanh 0.176. Set the indicator over 0.176 on the Th scale, read 0.174 on the C scale.

Verify that: tanh 0.236 = 0.232; tanh 0.528 = 0.484; tanh 1.145= 0.816

(b) Find x if tanh x = 0.372. Set indicator over 0.372 on the C scale. Read x =0.391 on the Th scale.

The Hyperbolic Cosine.

No special scale for the hyperbolic cosine is needed. From the definitions, tanh x = (sinh x)/(cosh x), and hence cosh x = (sinh x)/tanh x. This suggests the following rule:

Rule: **With C and D indices coinciding, set indicator over x on the Sh scale. Move slide until x on the Th scale is under the hairline. Read cosh x on the D scale under the C index.**

It will be observed that the first step in this rule sets the value of sinh x on the D scale. The second step sets the value of tanh x on the C scale in position for the division. The result of the division is then read on the D scale. Viewed in another way, sinh x may be multiplied by 1/tanh x. This reciprocal is automatically set on the CI scale in the second step of the rule above. For all values of x on the Sh scale, $1 < \cosh x \leqq 10.06$.

EXAMPLES:

(a) Find cosh 0.240. With the C and D scales coinciding, set the hairline over 0.240 on the upper Sh scale. Move the slide until 0.240 on Th is under the hairline. Read cosh 0.240 = 1.029 on the D scale at the C index.

(b) Find cosh 1.62. With C and D scales coinciding, set the hairline on 1.62 on the lower Sh scale. Move the slide until 1.62 on Th is under the hairline. Read cosh 1.62 = 2.63 on the D scale at the C index.

It follows from the definitions that $\cosh^2 x - \sinh^2 x = 1$, and hence $\sinh x = \sqrt{\cosh^2 x - 1}$. If the value of cosh x is given, and x is to be found, this formula may be used to convert the problem to the corresponding case for the hyperbolic sine.

EXAMPLE:

Given cosh x = 1.31, find x. Since $\sinh x = \sqrt{1.31^2 - 1} = \sqrt{1.716 - 1} = \sqrt{0.716} = 0.846$, when the hairline is set on 0.846 of the C scale, x = .768 may be read on the upper Sh scale.

COMPUTATIONS INVOLVING HYPERBOLIC FUNCTIONS

Computations involving hyperbolic functions are easily performed by usual methods (e.g., use of the C and D scales) by setting the values of the functions on the appropriate scales.

EXAMPLES:

(a) Find y = 24.6 sinh 0.35. Set the left index of the C scale opposite 24.6 on the D scale. Move indicator to 0.35 on the upper Sh scale. Read y = 8.79 on the D scale under the hairline.

(b) Find y = 86.4 tanh 0.416. Set the right index of C on 86.4 of D. Move indicator to 0.416 on Th. Read y = 34.0 on D under the hairline.

(c) Find $y = 77.3 \cosh 1.26$. In this case, it is best to set cosh 1.26 first. With C and D scales coinciding, set indicator on 1.26 of Sh. Move slide so that 1.26 of Th is under the hairline. Move indicator to 77.3 of the C scale. Read $y = 147.$ on D.

(d) Compute $17.9 \sinh 0.137 \times \sin 22°$. With C and D indices coinciding, set indicator on 0.317 of the upper Sh scale. Turn rule over and move slide so the right index of the C scale is under hairline. Move indicator to 22° on the S scale. Pull slide until 179 of the CI scale is under hairline. Read 2161 on D scale under the C index. The decimal point is found by noting that, approximately, $\sinh 0.317 = 0.3$, $\sin 22° = .4$, and hence $\sinh 0.317 \times \sin 22°$ is about 0.12 or ⅛. Then $17.9 \times$ ⅛ is about 2. Hence the result is 2.161.

Other Hyperbolic Functions

By definition, the following relations hold:

$1/\tanh x = \coth x$ ("the hyperbolic cotangent of x")

$1/\cosh x = \operatorname{sech} x$ ("the hyperbolic secant of x")

$1/\sinh x = \operatorname{csch} x$ ("the hyperbolic cosecant of x").

Since the values of these three additional functions are the reciprocals of functions discussed earlier, coth x and csch x may be read directly on the CI scale. After cosh x has been set on the D scale, sech x may be read on the DI scale on the front side of the rule, or if the indices coincide, on the CI scale of either side.

EXAMPLES:

Verify that $\coth 0.49 = 2.2$, $\operatorname{csch} 0.49 = 1.96$, $\operatorname{sech} 0.49 = 0.891$.

SMALL VALUES OF THE ARGUMENT

For small values of the argument x, the hyperbolic sine is approximately equal to x. Consequently, in computations involving sinh x for $x < 0.10$, no special scales are needed. The value of x may be set directly on a C or D or other appropriate scale and the computation continued. The same is true of the hyperbolic tangent. Moreover, the hyperbolic cosine for $x < 0.10$ is approximately equal to 1.

In evaluating hyperbolic functions of complex arguments, values of the circular sine and tangent less than 0.57° are sometimes needed. Although these values can be found by use of the special graduations for this purpose (see manual for Model No. 3, page 19), it is usually more convenient to read the ST scale as though the decimal point were at the left of the numbers printed, and to read the C (or D, CI, DI, etc.) scale with the decimal point one place to the left of where it would normally be. Thus $\sin 0.2° = 0.00349$; $\tan 0.16° = 0.00279$, read on the C scale.

EXAMPLES:

(a) Find $\sqrt{\sinh 0.073}$. Set the indicator over 73 on the D scale of the front side of the rule. Read 0.27 on the upper scale root scale. Then $\sqrt{\sinh 0.073} = 0.27$.

(b) Find $\log \tanh 0.06$. Set indicator over .06 on scale $1/N_4$. Read -1.222 on the D scale, or $8.778 - 10$ on the Co scale.

PART 3. HYPERBOLIC FUNCTIONS OF COMPLEX ARGUMENTS

The definitions of the hyperbolic functions may easily be extended to include cases in which the independent variables, or arguments, are complex numbers. Let z represent any complex number $x + jy$, where x and y are real numbers. Then:

$$\frac{e^z - e^{-z}}{2} = \sinh z, \quad \frac{e^z + e^{-z}}{2} = \cosh z, \quad \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh z.$$

By use of the definitions and the formula $e^{jz} = \cos z + j \sin z$ the following relations may be verified:

(1) $(e^{jz} - e^{-jz})/2j = \sin z$; $(e^{jz} + e^{-jz})/2 = \cos z$

(2) $\sinh z = - \sinh(-z) = -j \sin jz$

(3) $\cosh z = \cosh(-z) = \cos jz$

(4) $\sinh jz = j \sin z$

(5) $\cosh jz = \cos z$

(6) $\cosh^2 z - \sinh^2 z = 1$

(7) $\sinh z = \sinh(x + jy) = \sinh x \cosh jy + \cosh x \sinh jy$
$\quad = \sinh x \cos y + j \cosh x \sin y$

(8) $\cosh z = \cosh(x + jy) = \cosh x \cosh jy + \sinh x \sinh jy$
$\quad = \cosh x \cos y + j \sinh x \sin y$

(9) $\tanh z = \tanh(x + jy) = \sinh(x + jy)/\cosh(x + jy)$

(10) $\sin z = \sin(x + jy) = \sin x \cosh y + j \cos x \sinh y$
$\quad \cos z = \cos(x + jy) = \cos x \cosh y - j \sin x \sinh y$

(11) $\sinh(-z) = -\sinh z$
$\quad \cosh(-z) = \cosh z$
$\quad \tanh(-z) = -\tanh z$

For particular values of z each of these functions is, in general, a complex number which may be regarded as a vector expressible in either the component form or in exponential form, $\rho \underline{/\theta}$.

Sometimes the complex number z is given in exponential or polar form $\rho\underline{/\theta}$; for example, $\sinh \rho\underline{/\theta}$, or in particular, $\sinh 2.4\underline{/15°}$. In this case z may be expressed in the form $x + jy$ by means of the relations $x = \rho \cos \theta$, $y = \rho \sin \theta$. Thus $\sinh 2.4\underline{/15°} = \sinh\ 2.315 + j\,0.622$.

CHANGING SINH Z FROM COMPONENT TO EXPONENTIAL FORM

By formula (7) above, the complex number or vector

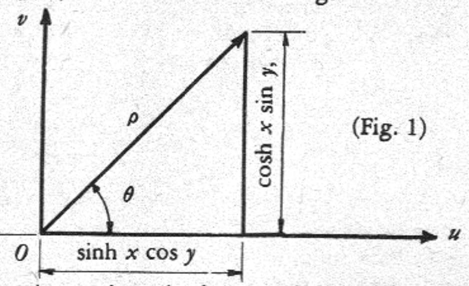
$$\sinh z = \sinh x \cos y + j \cosh x \sin y$$

is expressed in component form. If, for simplicity, u and v are defined by the formulas

$$u = \sinh x \cos y$$
$$v = \cosh x \sin y,$$

then $\sinh z = u + jv$.

A geometric representation of this complex number may be made by means of a u-axis of real numbers and a v-axis of pure imaginaries. The polar coordinates (ρ, θ) have their usual meanings.



(Fig. 1)

Methods of changing to the polar form $\rho/\underline{\theta}$ by use of the slide rule will now be explained. First, it should be noted that the real number y may be expressed in either radians or angular degrees. Since the graduations on the S and T scales are in terms of degrees, this measure is more convenient. A value of y given in radian measure should therefore first be converted to angular degrees.

To recall the formula for sinh z readily, notice the following analogies: For real variables,

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

is similar in *form* to

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

For the complex variable,

$$\sinh(x + jy) = \sinh x \cos y + j \cosh x \sin y,$$

there are *formal* similarities, but the operator j serves to replace the functions of y by ordinary *circular* functions. Thus, *x is always associated with hyperbolic functions, while y is associated with circular functions.*

The following relations (See Fig. 1) are basic to the computations:

I (a)
$$\tan \theta = \frac{v}{u} = \frac{\cosh x \sin y}{\sinh x \cos y} = \frac{\tan y}{\tanh x}$$

(b)
$$\rho = \frac{\sinh x \cos y}{\cos \theta}$$

Observe that the ratio for tan θ involves a function of y divided by a function of x, and is thus analogous to $\tan \theta = y/x$ for circular functions.

Finding ρ for sinh $(x+jy)$

Although it is usually better to find θ first, the explanations are long. The exposition is simplified by first considering a method of finding ρ assuming θ is known. If θ has been found first the value of ρ may be computed by the following rule, based on formula I (b).

Rule for ρ: **With C and D indices coinciding**
first, set the hairline over x on an Sh scale, and turn the rule over;
second, move the slide until θ on the S scale (read from *right to left* for the cosine) is under the hairline;
third, move the hairline to y on the S scale (read from *right to left*); read ρ on the D scale under the hairline.

Note that the first step sets sinh x on the C scale, the second step divides this by cos θ, and the third step multiplies by cos y. All the operations are actually done on the C and D scales, but only the final result needs to be read on D. The values of sinh x, cos y, and cos θ are automatically set.

EXAMPLES:

(a) Find ρ for sinh $(0.48 + j\,17°)$, given that $\theta = 34.4°$. With C and D indices together, set hairline over $x = 0.48$ on Sh. Move slide until 34.4 on S (reading from right to left) is under hairline. Move indicator to 17 on S (reading from right to left). Read $\rho = 0.578$ on D under hairline.

(b) Find ρ for sinh $(1.4 + j\,40°)$, given that $\theta = 43.4°$. With indices together, set indicator over 1.4 on Sh. Move slide until 43.4 on S (reading right to left) is under hairline. Move indicator to 40 on S (right to left). Read $\rho = 2.01$ on D under hairline.

(c) Find ρ for sinh $(0.73 + j\,2.2)$, given $\theta = 3.53°$. With C and D indices together, set hairline over $x = 0.73$ on Sh. The settings for cos 3.53 and cos 2.2 are so near the right end of the slide that practically no change from the original setting is observable. In other words, for $y = 2.2°$, the value of $v = \cosh x \sin y$ is near zero. The value of cos y is near 1, and ρ is approximately equal to $u = \sinh 0.73$. Hence $\rho = 0.797$.

Finding θ when $y < 45°$.

The ratio of the "pure imaginary" component to the real component determines the tangent of θ, and hence θ, as shown in formula I (a). The general rule is as follows.

Rule for θ: **To find θ for sinh $(x + jy)$ in the form $\rho/\underline{\theta}$:**
first, with C and D indices together, set hairline over y on a T scale;
second, move slide until x on Th is under hairline;
third, move indicator to C index;
fourth, move slide until C and D indices are together. Read θ on a T scale under the hairline.

The determination of the T scale on which θ is to be read depends upon the decimal point in the value of tan θ. This value may be noted on the D scale at the C index. However, to determine the decimal point, it is well to make a mental note of the approximate values of tan y and tanh x as they are set. By taking only the first digit, a mental computation easily gives the decimal point in the value of tan θ.

The following cases may arise. Note especially case (iii):

Rule:
(i) If $0.01 < \tan \theta \leq 0.1$, then $0.573° < \theta \leq 5.71°$ on ST
(ii) If $0.1 < \tan \theta \leq 1.0$, then $5.71° < \theta \leq 45°$ on T
(iii) If $1.0 < \tan \theta \leq 10.0$, then $45° < \theta \leq 84.3°$. In this exceptional case, the rule above must be altered. After step two, θ is on T above the right D index, and T is read from *right to left.*

The following cases may also occasionally occur:

(iv) If $10.0 < \tan \theta$, then $84.3° < \theta < 90°$. Since $\tan \varphi = \cot \theta$, where $\varphi = 90 - \theta$, the angle φ may be read on the ST scale and then $\theta = 90 - \varphi$.
(v) If $0 \leq \tan \theta \leq 0.01$, then $0 \leq \theta \leq 0.573°$. Read angle on ST, and divide by 10; that is, move decimal point one place to left of ST reading.

Step one sets the value of tan y on the D scale. Step two automatically sets the value of tanh x on the C scale for the division. The quotient is on the D scale at the C index. Since this is the value of tan θ, when the indices are brought together in step four the value of θ on a T scale is under the hairline. If desired, the following can be substituted for steps three and four above.

Third, read D scale at C index and move indicator over this value on C.

Fourth, read θ on a T scale under the hairline.

Although these last two rules may appear easier to use than the others, it should be remembered that to start finding ρ by the rule given earlier the indices must be together. Thus, the rule as originally given ends with the slide in position to begin finding ρ. Other methods are given later.

Since tanh $x < 1$, it follows from the relation tan θ = tan y/tanh x that $\theta > y$ for all values of $y < 90°$. Moreover, since tanh $x \rightarrow 1$ as x becomes larger, the difference $\theta - y$ becomes smaller as x increases. These observations are sometimes useful as a rough check on θ. Thus, for sinh $(1 + j\,75°)$ the value of θ is 78.45° (Note 78.45 > 75); for sinh $(2 + j\,75°)$, $\theta = 75.5$; for sinh $(1 + j\,81°)$, $\theta = 83.15$, and for sinh $(2 + j\,81°)$, $\theta = 81.34°$.

Each of the following examples should be followed through several times to gain familiarity with the method and to observe how quickly the calculation can be completed when the details of the explanation are omitted.

Examples:

(a) Find θ for sinh $(0.48 + j\,17°)$. In this case, tan θ = tan 17°/ tanh 0.48. Set the slide so that C and D scales coincide. Move indicator to 17 on T. Note on the D scale that tan 17 is about 0.3. Turn rule over and move slide until $x = 0.48$ on Th is under hairline. Note on the C scale that tanh 0.48 is about 0.4, and hence tan θ is roughly 0.3/0.4 = ¾ = .75. Actually, it is 0.685, read on the D scale at the C index. This is case (ii). Move the indicator to the C index, bring the C and D indices together, and read $\theta = 34.4$ on the T scale under the hairline. The value of $\rho = 0.578$ was found in Example (a) page 11. Hence $\rho\,\underline{/\theta} = 0.578\,\underline{/34.4°}$.

(b) Find θ for sinh $(0.73 + j\,2.2°)$. Here tan θ = tan 2.2/tanh 0.73. With C and D scales coinciding, set hairline over 2.2° on ST. Observe that tan 2.2° = 0.04, approximately. Turn rule over, and move slide so 0.73 on Th is under hairline. Note tanh 0.73 = 0.6+ on the C scale. Hence tan θ = 0.04/0.6 roughly, or 0.0616, which can be read on the D scale at C index. This is case (i). Move indicator to C-index, and bring indices together. Read $\theta = 3.53°$ on ST. The value of ρ was found in Example (c) page 11. Hence $\rho\,\underline{/\theta} = 0.797\,\underline{/3.53°}$.

(c) Find $\rho\,\underline{/\theta}$ for sinh $(0.55 + j\,1.5°)$. Make C and D indices coincide, move hairline to 1,5° on ST. Note tan 1.5° is roughly 0.025 (on C or D). Move slide so 0.55 on Th is under hairline. Note tanh (0.55) is 0.5 on C. Hence tan θ = 0.025/0.5, or tan θ is about 0.05 (on D at C index). This is case (i). Move indicator to C index, bring indices together, and read $\theta = 3°$ on ST under hairline. In this case, ρ is approximately equal to sinh 0.55, or $\rho \doteq 0.58$. Hence $\rho\,\underline{/\theta} = 0.58\,\underline{/3°}$.

(d) Find $\rho\,\underline{/\theta}$ for sinh $(0.19 + j\,4.2°)$. With C and D scales together, set hairline on 4.2° of ST. Move slide so that 0.19 on Th is under hairline. Observe that, roughly, tan θ = 0.07/0.19 or about 0.4. This is case (ii). Move indicator to C index, and bring indices of C and D to coincidence.

Read $\theta = 21.36°$ on T under hairline. Set hairline over 0.19 on Sh. Move slide until 21.36° on S (read right to left) is under hairline. Move slide to 4.2° near right index of S. Read $\rho = 0.201$ on D under hairline. Hence sinh $(0.19 + j\,4.2°) = 0.201\,\underline{/21.36°}$.

(e) Find $\rho\,\underline{/\theta}$ for sinh $(0.424 + j\,38°)$. With C and D indices together set hairline over 38° on T. Note tan 38° is about 0.8. Move slide until 0.424 on Th is under hairline. Note on C that tanh 0.424 is about 0.4. Then tan θ = 1.95 (on D at C index). This is case (iii). Since 1.95 on CI is at the right D index, move indicator to index and read $\theta = 62.88°$ on T. To find ρ, set the value of sinh 0.424 on D, move slide to bring cos 62.88° under hairline, move indicator to cos 38° on S. Read $\rho = 0.756$ on D. Hence $\rho\,\underline{/\theta} = 0.756\,\underline{/62.88°}$.

Finding θ when $y > 45°$

When $y > 45°$ the value of tan y is found on the CI scale. The formula I(a) takes the form:

$$\text{I(c)} \qquad \tan\theta = \frac{1}{\tanh x \cot y},$$

and this suggests the use of the DI scale for the multiplication.

Rule: **To find θ for sinh $(x + jy)$ in the form ρ/θ when $y > 45°$:**

first, with C and D indices together, set the indicator over x on the Th scale.

Second, move the slide so the C index is under the hairline. Move indicator to y on T, reading from *right to left*. Observe that the value of tan θ may be read on DI and the value of cot θ on D.

Third, bring C and D indices together. If tan $\theta < 10$, read θ on T. If tan $\theta > 10$, read φ on ST, and compute $\theta = 90 - \varphi$.

In this case, the order of the first operations may be interchanged. That is, the value of tan y may be set first on DI, and then multiplied by tanh x, but the order given in the rule is more convenient.

Examples:

(a) Find θ for sinh $(0.31 + j\,58°)$. With C and D scales together, set indicator on 0.31 of Th. Move right C-index under hairline, then move indicator to 58° on T. Note on DI that tan $\theta = 5.34$. Move slide until C and D indices coincide, and read $\theta = 79.37°$ on T. With C and D indices together, set hairline on 0.31 of Sh, pull slide until 79.37 on S is under hairline, move indicator to 58° on S, read $\rho = 0.905$ on D. Hence, sinh $(0.31 + j\,58°) = 0.905\,\underline{/79.37°}$.

(b) Find θ for sinh $(0.31 + j\,75°)$. With C and D scales together, set hairline on 0.31 of Th. Move left C-index under hairline, then move indicator to 75° on T. Note that tan θ = 12.4 + on DI. This is case (iv). Bring C and D scales together, and then read $\varphi = 4.6°$ on ST. Hence $\theta = 90° - 4.6° = 85.4°$. To find ρ, with indices together set $x = 0.31$ on Sh, move slide until 4.6 on ST is under hairline, then move indicator to 75° on S (read right to left). Read $\rho = 1.014$ on D. Hence, $\rho\,\underline{/\theta} = 1.014\,\underline{/85.4°}$.

(c) Find θ for sinh $(1.4 + j\,80°)$. With C and D scales together, set hairline on 1.4 of Th. Move right C-index under hairline, then move indicator to 80° on T. Move slide to bring indices together, and read $\theta = 81.1$ on T; or note 6.4 under hairline on DI, and instead of moving slide, move indicator to 6.4 on CI, then read θ on T. With indices together.

divide sinh 1.4 by cos 81.1; multiply by cos 80; read $\rho = 2.14$. Hence $\rho\,\underline{/\theta} = 2.14\,\underline{/81.1^\circ}$.

ALTERNATIVE METHODS FOR SINH *Z*

The alternative methods of treating sinh z described below have certain advantages and also certain disadvantages. If the methods outlined above are used, the following will serve as checks.

(i) The formulas $u = \sinh x \cos y$; $v = \cosh x \sin y$ may be used to compute u and v. Then $\sinh z = u + jv$ may be converted to exponential form by ordinary vector methods. In this case, three major steps are required.

EXAMPLE:

For $\sinh(0.48 + j\,17^\circ)$, $u = 0.477$, $v = 0.327$; $u + jv = 0.477 + j\,0.327$. Using the method described in Part 1, $\theta = 34.4^\circ$, $\rho = 0.578$.

(ii) The formula I (a), $\tan\theta = \tan y/\tanh x$, suggests that θ may be obtained from the complex number

(12) $r\underline{/\theta} = \tanh x + j\tan y$, although in general r is not equal to ρ (See Fig. 2).

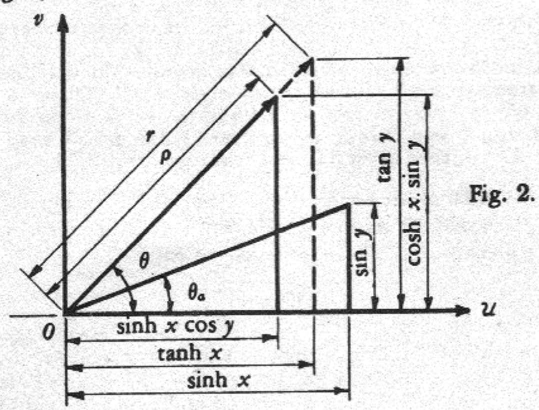


Fig. 2.

Moreover, the relation

(13) $\rho = \sqrt{\sinh^2 x + \sin^2 y}$

suggests that ρ may be obtained from the complex number

(14) $\rho\underline{/\theta_a} = \sinh x + j\sin y$, where $\tan\theta_a = \sin y/\sinh x$. It follows from I (a) that $\tan\theta = \dfrac{\cosh x}{\cos y}\tan\theta_a$.

Since, for all values of $x \neq 0$ and of $y \neq 0$, $(\cosh x) > 1$ and $(\cos y) < 1$, the ratio $(\tan\theta/\tan\theta_a) > 1$ and consequently $\theta > \theta_a$. From Fig. 2,

(15) $\cos\theta = \dfrac{\sinh x \cos y}{\rho} = \dfrac{\tanh x}{r}$,

and therefore $\rho = r\cosh x\cos y$. These results lead to the following methods.

The value of ρ may be obtained from formula (14) by ordinary vector methods (as explained in manual for Model No. 3) and then θ found by use of (15).

EXAMPLE:

Let $\sinh(0.48 + j\,17^\circ) = \rho\,\underline{/\theta}$. Then

$$\begin{aligned}\rho\,\underline{/\theta_a} &= \sinh 0.48 + j\sin 17^\circ\\ &= 0.4986 + j\,0.292\\ &= 0.578\,\underline{/30.4^\circ}\ \text{(Note: } \theta_a = 30.4 < \theta).\end{aligned}$$

This calculation is most readily carried out if the slide rule is equipped with *two* indicators. With C and D indices together, set one indicator on 0.48 on Sh, the other on 17° of S. Move C index to the larger reading on C, read $\theta_a = 30.4^\circ$ on T under the hairline. Then move slide until 30.4° on S is under this hairline. Read ρ on D at C index. Using formula (15), since ρ is known, find $\theta = 34.4^\circ$.

(iii) The angle θ may be found by ordinary vector methods from formula (12), and then ρ found by I (b), or (15) as explained earlier.

EXAMPLE:

For $\sinh(0.48 + j\,17^\circ)$, $r\underline{/\theta} = \tanh 0.48 + j\tan 17^\circ$. With indices together, set right indicator on 0.48 of Th, and left on 17° of T. Move C index to larger reading, and read $\theta = 34.4$ under other hairline. Find ρ by regular methods.

(iv) The value of ρ may be obtained from (13) and then θ found by use of (12) or (15).

In using any method, it is wise to give some attention to the approximate values being set on the scales, and to the operations being performed as suggested by the formulas. In the long run, it is probably best to adopt one method and use it almost exclusively, rather than to invite confusion and error by attempting a variety of methods.

CHANGING COSH *Z* FROM COMPONENT TO EXPONENTIAL FORM

By formula (8) above, the complex number or vector

$\cosh z = \cosh x\cos y + j\sinh x\sin y$.

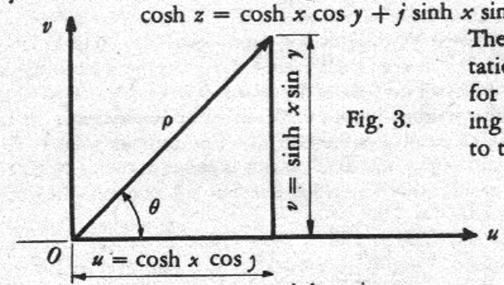


Fig. 3.

The geometric representation is similar to that for sinh z. The following relations are basic to the computations:

II (a) $\quad \tan\theta = \dfrac{v}{u} = \dfrac{\sinh x\sin y}{\cosh x\cos y} = \tanh x\tan y$

(b) $\quad \rho = \dfrac{\sinh x\sin y}{\sin\theta}$

Note that I (b) for sinh z expresses ρ in terms of sinh x and the *cosines* of y and θ, while II (b) for cosh z expresses ρ in terms of sinh x and the *sines* of y and θ. Note also that $\tan\theta$ above could be written $\tan y/\coth x$, but that this form is less convenient for computation. It should be noticed that, since $(\tanh x) < 1$, it follows that $\theta < y$ for the hyperbolic cosine.

Finding ρ for Cosh $(x + jy)$.

Assume θ has been found. The rule for ρ is the same as given earlier (page 10) for sinh z except the S scale is read from *left to right* for sines.

EXAMPLES:

(a) Find ρ for cosh $(0.31 + j\,58°)$ if $\theta = 25.7°$. With C and D indices together, set hairline on $x = 0.31$ of Sh. Move slide until 25.7° on S is under hairline, then move indicator to 58° on S. Read $\rho = 0.616$ on D under hairline.

(b) Find ρ for cosh $(1.4 + j\,80°)$ if $\theta = 78.7°$. Set sinh 1.4 on D at 1.904. Move slide so 78.7° on S is under hairline, then indicator to 80° on S. Read $\rho = 1.91$ on D. Notice that y and θ are so nearly equal that the ratio of their sines is near 1, and hence ρ is approximately equal to sinh 1.4.

Finding θ for Cosh $(x + jy)$ when $y < 45°$.

Formula (IIa) shows that tan θ can be found by a simple multiplication using the Th and T or ST scales with the D scale. The rules for determining on which scale (T or ST) the angle θ may be found are the same as those given earlier (page 11). However, since tanh $x < 1$ and, for $y < 45°$, tan $y < 1$, it follows that tan $\theta < 1$, and cases (iii) and (iv) cannot arise for $y < 45°$.

Rule for θ when $y < 45°$.

With C and D indices together, set indicator on x of Th. Move slide until an index of C (right or left as needed) is under hairline. Move indicator to y on T (or ST). Bring indices together, and read θ on T or ST.

EXAMPLES:

(a) Find ρ and θ for cosh $(0.23 + j\,16°)$. Here $\tan\theta = (\tanh 0.23) \times (\tan 16°)$. With C and D indices together, set indicator over 0.23 on Th. Move slide so left C index is under hairline. Move indicator to 16° on T. Since $\tan\theta = (0.2)(0.3) = 0.06$, roughly, or, more accurately, 0.0648, this is case (i), and $\theta = 3.71°$ + is read on ST after the scales have been made to coincide. Now set indicator on 0.23 of Sh (about 0.23 on D), move slide until 3.71° on ST is under hairline (about 0.064 on C), move indicator to 16 on S (about 0.276 on C). Read $\rho = 0.99$ on D. The decimal point is determined by mentally calculating 0.23/0.06, which is about 4; then 4×0.276 is about 1. Hence the result cannot be either 0.099 or 9.9, and must be 0.99. Thus, cosh $= (0.23 + j\,16°) = 0.99\,\underline{/3.71°}$.

(b) Find ρ and θ for cosh $(0.68 + j\,40°)$. With C and D indices together, set $x = 0.68$ on Th. Note (roughly) 0.6 on D. Turn rule over and move right C index under hairline. Move indicator to 40° on T. Note 0.8 + on C, and the product $(0.6)(0.8) = 0.496$ on D. This is Case (ii). Bring indices together and read $\theta = 26.4°$ on T. Set indicator over 0.68 on Sh. Move slide until 26.4 on S is under hairline. (Note quotient sinh 0.68/sin 26.4° is about 1.65). Multiply by sin 40°; reading 1.06 on D. Hence $\rho\,\underline{/\theta} = 1.06\,\underline{/26.4°}$.

(c) Find ρ and θ for cosh $(0.285 + j\,4.2°)$. With C and D indices together, set $x = 0.285$ on Th. Set right C index under hairline, then move indicator to 4.2° on ST. Note $.3 \times .07 = .021$, and hence this is case (i). With scales coinciding, read $\theta = 1.165°$ on ST. Calculate $\rho = 1.046$. Hence $\rho\,\underline{/\theta} = 1.046\,\underline{/1.165°}$.

(d) Find $\rho\,\underline{/\theta}$ for cosh $(0.2 + j\,1.3°)$. Note tan 0.2 is about 0.2; tan 1.3° is about 0.02. Hence tan θ is about 0.004, on scale D. This is case (v). When indices are together, read $\theta = 0.257°$ on ST with decimal point moved one place to the left. In this case, $\rho = 1.02$ approximately.

Finding θ for Cosh $(x + jy)$ when $y > 45°$.

When $y > 45°$, $(\tan y) > 1$, and tan y is most easily read on the CI scale.

***Rule** for θ **when** $y > 45°$.*

With C and D indices together, set indicator on x of Th. Move slide until y on T (read from *right to left*) is under hairline. (Note value of tan y on CI, and tan θ on D at C index.) Read θ on the T scale determined by the value of tan θ.

EXAMPLES:

(a) Find $\rho\,\underline{/\theta}$ for cosh $(0.31 + j\,58°)$. With C and D indices together, set hairline on $x = 0.31$ on Th. Note tanh $x = 0.3$. Move slide until 58° on T is under hairline. Note tan $58 = 1.6$ on C, and $\tan\theta = 0.48$ on D at C index. This is case (ii). Bring scales to coincidence and read $\theta = 25.7°$ on T. Since $\rho = 0.616$ for this problem was found in Example (a), page 16, $\rho\,\underline{/\theta} = 0.616\,\underline{/25.7°}$. Notice $\theta < y$ as a very rough check.

(b) Find $\rho\,\underline{/\theta}$ for cosh $(1.4 + j\,80°)$. Set tanh 1.4 on D. Notice the value is about 0.9. To multiply by tan 80°, move slide so 80° on T is under hairline. Notice tan 80° is near 6 on CI. Hence $\tan\theta = 5.02$ on D at C index. This is case (iii). Move indicator to 5.02 on CI (at the D index), and read $\theta = 78.7°$ on T. Note $\theta < y$. Since $\rho = 1.91$ was found in Example (b), page 16, $\rho\,\underline{/\theta} = 1.91\,\underline{/78.7°}$.

(c) Find $\rho\,\underline{/\theta}$ for cosh $(2 + j\,87°)$. In this case, $y > 84.3°$. Hence $\tan 87° = \cot(90\text{-}87) = \cot 3°$, or 19.1 read on CI opposite 3° on ST. Thus set tanh 2 on D, move slide to 3° on ST. Hence $\tan\theta = 18.4$, and this is case (iv). Read $\varphi = 3.11°$ on ST at the D index, and thus $\theta = 90 - 3.11 = 86.89°$, which is, as it should be, less than $y = 87°$. To find ρ, set sinh 2 = 3.63 on D. Since θ and y are approximately equal, it follows that $\rho = 3.63$ approximately. Hence $\rho\,\underline{/\theta} = 3.63\,\underline{/86.9°}$.

ALTERNATIVE METHODS FOR COSH $(x + jy)$

Other methods of computing ρ and θ for cosh $(x + jy) = u + jv$ are briefly outlined below:

(i) Compute numerical values of $u = \cosh x \cos y$ and $v = \sinh x \sin y$. Convert the complex number $u + jv$ to exponential form by ordinary vector methods.

(ii) The formula (IIa) written in the form

(16) $\tan\theta = \tanh x \cot y = \tan y/\coth x$

suggests that θ may be obtained from either of the complex numbers

(17) $r_a\,\underline{/\theta} = \cot y + j \tanh x$, or $r_b\,\underline{/\theta} = \coth x + j \tan y$. Although in general $r_a \neq r_b \neq \rho$. Moreover, the relation

(18) $\rho = \sqrt{\sinh^2 x + \cos^2 y}$,

which may be verified by computing $\rho = \sqrt{u^2 + v^2}$ and using formula (6), suggests that ρ may be found from the complex number

(19) $\rho\,\underline{/\theta_a} = \sinh x + j \cos y$.

The θ_a in this formula satisfies the relation

$$\tan\theta = \frac{\sin y}{\cosh x} \cot\theta_a, \text{ and } \rho = r_a \cosh x \sin y.$$

By these and other possible formulas the problem may be reduced to one which may be solved by ordinary vector methods as described earlier.

EXAMPLE:

Let cosh $(0.31 + j\,58°) = \rho\,\underline{/\theta}$; then $u = \cosh 0.31 \cos 58° = 0.556$; $v = \sinh 0.31 \sin 58 = 0.267$. Hence $u + jv = 0.616 + j\,25.7°$, by methods described in Part I. If $r_a\,\underline{/\theta} = \cot 58 + j \tanh 0.31$, set cot 58° on D by placing indicator over 58° on T. Move right C− index to hairline, then move indicator to tanh 0.31 = 0.3 on D. Read $\theta = 25.7°$ on T; here $r_a = 0.693$. If $\rho\,\underline{/\theta_a} = \sinh 0.31 + j \cos 58°$, set right index of C on cos 58° = 0.53. Set indicator over sinh 0.31 = 0.315 on D. Read $\theta_a = 30.7°$ on T. Move slide so 30.7° on S is under hairline, read $\rho = 0.616$ on D at C index.

CHANGING TANH Z FROM COMPONENT TO EXPONENTIAL FORM

Although tanh $(x + jy)$ can be expressed in the form $u + jv$, the formulas for u and v are not convenient for slide rule work. However, $\tanh z = \sinh z/\cosh z$ is expressible as a quotient. If the complex numbers sinh z and cosh z are expressed in the exponential form

$$\sinh z = \rho_1\,\underline{/\theta_1}, \text{ and } \cosh z = \rho_2\,\underline{/\theta_2}, \text{ then}$$

$$\tanh z = (\rho_1/\rho_2)\,\underline{/\theta_1 - \theta_2}$$

Rule for Tanh z.

Express sinh z in the form ρ_1/θ_1.
Express cosh z in the form ρ_2/θ_2.
Then tanh $z = (\rho_1/\rho_2)\,\underline{/\theta_1 - \theta_2}$.

EXAMPLES:

(a) Find $\rho\,\underline{/\theta}$ for tanh $(0.31 + j\,58°)$.
From Example (a), page 13, sinh $(0.31 + j\,58°) = 0.905\,\underline{/79.37°}$
From Example (a), page 17, cosh $(0.31 + j\,58°) = 0.616\,\underline{/25.7°}$

$$\text{Hence } \tanh (0.31 + j\,58°) = \frac{0.905}{0.616}\,\underline{/79.37° - 25.7°}$$
$$= 1.47\,\underline{/53.67°}$$

(b) Find $\rho\,\underline{/\theta}$ for tanh $(1.4 + j\,80°)$.
From Example (c), page 13, sinh $(1.4 + j\,80°) = 2.14\,\underline{/81.1°}$
From Example (b), page 17, cosh $(1.4 + j\,80°) = 1.91\,\underline{/78.7°}$

$$\text{Hence } \tanh (1.4 + j\,80°) = \frac{2.14}{1.91}\,\underline{/81.1° - 78.7°}$$
$$= 1.12\,\underline{/2.4°}$$

ILLUSTRATIVE APPLIED PROBLEMS

1. Determine the "insertion loss" caused by inserting a line between a generator and a load using the formula

$$\frac{I_1}{I_2} = \cosh \gamma l + \frac{Z_g Z_r + Z_0^2}{(Z_g + Z_r) Z_0} \sinh \gamma l,$$

when $Z_g = 215$ ohms, $Z_r = 420$ ohms, $l = 150$ miles, $Z_0 = 720\,\underline{/-15°}$, $\alpha = 0.00720$ neper/mile, $\beta = 0.0280$ radian/mile, and $\gamma l = l\,(\alpha + j\beta)$. First find

$$\gamma l = 150\,(0.00720 + j\,0.0280) = 1.08 + j\,4.20 = 1.08 + j\,240.5°$$

Since cosh γl is to be added to another complex expression, express it in component form.

$$\cosh (1.08 + j\,240.5°) = \cosh 1.08 \cos 240.5° + j \sinh 1.08 \sin 240.5°$$
$$= -\cosh 1.08 \cos 60.5° - j \sinh 1.08 \sin 60.5°$$

With indices together, set indicator on 1.08 of Sh. Move slide until 1.08 on Th is under hairline. Exchange indices, and move hairline over 60.5° on S (reading right to left). Find 0.809 on D. Then set sinh 1.08 = 1.30 + on the D scale and multiply by sin 60.5° using the S scale. Read 1.13 on D. Hence cosh $(1.08 + j\,240.5°) = -0.809 - j\,1.13$.

As an illustration of another method of finding cosh $(1.08 + j\,240.5°)$, take $y = 240.5° - 180° = 60.5°$, set indices together and indicator over 1.08 on Th. Move slide until 60.5° on T is under hairline, observe that $\tan \theta = 1.4$, and read 54.5° on T at right index of D. With indices together again set hairline on 1.08 of Sh, move slide until 54.5° on S is under hairline, then move hairline to 60.5° on S. Read $\rho = 1.393$ on D under hairline. Hence

$$\cosh (1.08 + j\,240.5°) = 1.393\,\underline{/180° + 54.5°} = 1.393\,\underline{/234.5°}$$
$$= 0.809 - j\,1.13 +.$$

Next compute the coefficient of sinh γl.

$$\frac{Z_g Z_r + Z_0^2}{Z_0\,(Z_g + Z_r)} = \frac{215 \times 420 + (720\,\underline{/-15°})^2}{(215 + 420)\,720\,\underline{/-15°}},$$

$$= \frac{90{,}300 + 518{,}000\,\underline{/-30°}}{457{,}000\,\underline{/-15°}},$$

$$= \frac{90{,}300 + 449{,}000 - j\,259{,}000}{457{,}000\,\underline{/-15°}}$$

$$= \frac{539{,}000 - j\,259{,}000}{457{,}000\,\underline{/-15°}}$$

$$= \frac{598{,}000\,\underline{/-25.65°}}{457{,}000\,\underline{/-15°}}$$

$$= 1.308\,\underline{/-10.65°}$$

The coefficient just computed is to be multiplied by sinh γl, and hence it is best to express the latter in polar form. Except for algebraic signs, sinh $(1.08 + j\,240.5°) = \sinh (1.08 + j\,60.5°)$. With indices coinciding, set hairline on 1.08 of Th. Move right C-index to hairline, turn the rule over, and move the indicator to 60.5° on T. Set hairline over 1.08 of Sh. Move slide until 65.8° on S (reading right to left) is under hairline, then move indicator to 60.5° on S. Read $\rho = 1.565$ on D. Since the original angle of 240.5° was in the third quadrant, the angle found must be corrected by adding 180°. Then sinh $(1.08 + j\,240.5°) = 1.565\,\underline{/245.8°}$

Multiplying the two factors gives

$$1.308\,\underline{/-10.65°} \times 1.565\,\underline{/245.8°} = 2.05\,\underline{/235.2°}$$

In order to add this to the first term in the formula, the polar form should be replaced by the component form.

Compute 235.2° − 180° = 55.2°, and then 2.05 cos 55.2 = 1.17 and 2.05 sin 55.2° = 1.68. Both results must be given a negative sign since 235.2° is in the third quadrant. Thus

$$\begin{aligned} I_1/I_2 &= -0.809 - j\,1.13 + (-1.17 - j\,1.68) \\ &= -1.979 - j\,2.81 \end{aligned}$$

This may be converted to polar form, noting that the angle is in the third quadrant. Thus

$$\begin{aligned} I_1/I_2 &= 3.43 \underline{/180° + 54.8} \\ &= 3.43 \underline{/234.8°}, \end{aligned}$$

approximately, and the absolute value is 3.43. Finally, compute $\log_e 3.43$ by setting 3.43 on scale $1/N_3$ and reading on DF the value 1.233, the "insertion loss" in nepers.

Although not a part of this problem, tanh (1.08 + j 240.5°) is easily obtainable from the above results and will be found to illustrate the method. Thus

$$\frac{\sinh (1.08 + j\,240.5°)}{\cosh (1.08 + j\,240.5°)} = \frac{1.565 \underline{/245.8°}}{1.393 \underline{/234.5°}} = 1.122 \underline{/11.3°}$$

It will prove to be interesting, and good practice in computation with the hyperbolic scales, to carry through the calculations for l = 100 miles and l = 200 miles, and to compare the results with those for l = 150 miles given above. Below are some results typical of those obtainable quickly without careful attention to accuracy.

If l = 100 mi., then $\gamma l = 0.72 + j\,2.8 \doteq 0.72 + j\,160°$
$\cosh (0.72 + j\,160°) = -1.193 + j\,0.268$, and
$\sinh (0.72 + j\,160°) = 0.855 \underline{/180° - 30.5°} = 0.855 \underline{/149.5°}$

The coefficient $1.308 \underline{/-10.65°}$ is unchanged, and
$1.308 \underline{/-10.6°} \times 0.855 \underline{/149.5°} = 1.118 \underline{/138.9°}$,
and this is equal to $-0.843 + j\,0.734$. Hence

$$\begin{aligned} I_1/I_2 &= 1.193 + j\,0.268 + (-0.843 + j\,0.734) \\ &= -2.036 + j\,1.002 = 2.27 \underline{/180° - 26.2°}. \end{aligned}$$

Then $\log_e 2.27 = 0.818$ nepers loss.

If l = 200 miles, $\gamma l = 1.44 + j\,320° = 1.44 - j\,40°$;
$\cosh (1.44 - j\,40°) = 1.71 - j\,1.28$;
$\sinh (1.44 - j\,40°) = 2.09 \underline{/-43.2°}$.

Then $1.308 \underline{/-10.6°} \times 2.09 \underline{/-43.2°} = 2.735 \underline{/-53.8°} \doteq 1.61 - j2.21$.
Hence $I_1/I_2 = 3.32 - j3.49 = 4.82 \underline{/-46.4°}$, and $\log_e 4.82 = 1.57$ nepers loss.

2. Determine the voltage $V_{ro} = \dfrac{V_s}{\cosh S\sqrt{zy}}$, at the end of an open circuit line in which the sending end voltage V_s = 4 volts, the length of the line S = 50 miles, and $\sqrt{zy} = \alpha + j\beta$, where α = 0.0048 neper/mile and β = 0.0276 radian/mile. Thus

$$V_{ro} = \frac{4\underline{/0°}}{\cosh (0.24 + j\,1.38)}$$

First change 1.38 radians to degrees. Thus set π on CF opposite 180 on DF, move hairline to 1.38 on CF, and read 79° on DF under the hairline. To find cosh (0.24 + j 79°), first set indices together, and then set hairline over 0.24 on Th. Move slide until 79° on T is under hairline. Note $\tan\theta = 1.21+$, read $\theta = 50.5°$ on T at right index.

With C and D indices together, set hairline on 0.24 of Sh. Move slide until 50.5° on S is under hairline, then set hairline on 79° of S. Read $\rho = 0.308$ on D. Then

$$V_{ro} = \frac{4 \underline{/0°}}{0.308 \underline{/50.5°}} = 13 \underline{/-50.5°} \text{ volts, approximately.}$$

PART 4. INVERSE HYPERBOLIC FUNCTIONS OF COMPLEX ARGUMENTS

If ρ and θ are known, and $z = x + jy$ is to be determined so that $\sinh z = \rho\underline{/\theta}$, an inverse problem is involved. Similar problems arise involving cosh z and tanh z.

Finding z for sinh z = ρ/θ.

To find $z = x + jy$ when $\sinh z = \rho\underline{/\theta}$, or $z = \text{arc}\sinh \rho\underline{/\theta}$, first write $\sinh z = \rho \cos\theta + j\rho \sin\theta = u + jv$ where $u = \rho\cos\theta$ and $v = \rho\sin\theta$ are known numbers. Since

$\sinh(x + jy) = \sinh x \cos y + j \cosh x \sin y$, equating the real and the imaginary components in the two expressions for sinh z, one has the equations

$$\begin{aligned} \sinh x \cos y &= u \\ \cosh x \sin y &= v, \end{aligned} \tag{20}$$

from which x and y are to be found. Now by use of equation (6),
$v^2 = (1 + \sinh^2 x)\sin^2 y$, and from (20), $\sinh^2 x = u^2/\cos^2 y$.
On eliminating x and replacing $\cos^2 y$ by $1 - \sin^2 y$, the equation

$$\sin^4 y - (u^2 + v^2 + 1)\sin^2 y + v^2 = 0$$

is obtained. This is a quadratic in $\sin^2 y$, and hence

$$\sin^2 y = \left\{(u^2 + v^2 + 1) \pm \sqrt{(u^2 + v^2 + 1)^2 - 4v^2}\right\}/2,$$

in which the positive sign must be discarded. Then $\sqrt{2}\sin y = \sqrt{p - \sqrt{q}}$, where $p = u^2 + v^2 + 1$ and $q = p^2 - 4v^2$. In algebra it is shown that an irrational expression of this form can be written in the simpler form $\sqrt{U_1} \pm \sqrt{V_1}$ provided $p^2 - q$ is a perfect square. Since in this case $p^2 - q = 4v^2$, the condition is fulfilled. Thus U_1 and V_1 are to be found such that

$$\sqrt{p \pm \sqrt{q}} = \sqrt{U_1} \pm \sqrt{V_1}$$

Squaring, and equating first the rational and then irrational parts, on solving the two equations thus obtained one has

$$\begin{aligned} U_1 &= \{u^2 + (v+1)^2\}/2; \\ V_1 &= \{u^2 + (v-1)^2\}/2. \end{aligned}$$

Hence it follows that

III (a) $$\sin y = \frac{U_s - V_s}{2}, \text{ where}$$

$$U_s = \sqrt{u^2 + (v+1)^2} \text{ and } V_s = \sqrt{u^2 + (v-1)^2},$$

from which y can be found in terms of u and v. The negative sign is chosen because otherwise the value of sin y would exceed 1. From equations (20)

III (b) $$\sinh x = u/\cos y,$$

from which x may be found.

It is convenient to regard U_s as the resultant of a vector whose components are u and $v + 1$; and similarly, to regard V_s as the resultant of a vector whose components are u and $v - 1$.

Then for convenience writing

$$U_s \angle\theta_a = u + j\,(v+1) \text{ and } V_s \angle\theta_b = u + j\,(v-1),$$

one may compute U_s and V_s by the methods of Part I.

The example which follows is the inverse of one solved earlier in Part III in order that the meaning of the process may be clarified.

Example.

Find $z = x + jy$ so that $\sinh z = 0.201 \angle 21.36°$
First find $u = 0.201 \cos 21.36 = 0.187$, and
$v = 0.201 \sin 21.36 = 0.0734$.
Then $U_s \angle\theta_a = 0.187 + j\,1.074 = 1.09 \angle 81°+$ or $U_s = 1.09$;
$V_s \angle\theta_b = 0.187 + j\,0.9266 = 0.945 \angle 78.6° -$ or $V_s = 0.945$

Therefore $\sin y = \dfrac{1.09 - 0.945}{2} = .0725$, and hence $y = 4.15°$, which is found by setting .0725 on D and reading y on ST. To find x, find $\sinh x = 0.187/\cos 4.15° = 0.187/0.997 = 0.188$ or approximately 0.19. Hence $x = 0.19$ and $y = 4.2$ are the approximate values of x and y. Compare with Example (d), page 12.

Finding z for cosh z = ρ/θ.

To find $z = x + jy = \text{arccosh}\, \rho \angle\theta$, where ρ and θ are known, first find $u = \rho \cos\theta$ and $v = \rho \sin\theta$. Since

$$\cosh z = \cosh x \cos y + j \sinh x \sin y,$$

the equations

$$\cosh x \cos y = u$$
$$\sinh x \sin y = v$$

are to be solved for x and y. In this case it turns out to be more convenient to solve for cos y by the method described for sin y above. The results are as follows:

IV (a) $$\cos y = \frac{U_c - V_c}{2}, \text{ where}$$

$$U_c = \sqrt{(u+1)^2 + v^2} \text{ and } V_c = \sqrt{(u-1)^2 + v^2}.$$

(b) $$\sinh x = v/\sin y$$

Thus cosh z may be treated by methods essentially similar to those for sinh z outlined above.

Example:

If $\cosh z = 1.06 \angle 26.4°$, find $z = x + jy$.
First compute $u = 1.06 \cos 26.4° = 0.949$ and
$v = 1.06 \sin 26.4° = 0.472$. Then
$U_c = \sqrt{1.949^2 + 0.472^2}$ or 2.01
$V_c = \sqrt{0.051^2 + 0.472^2}$ or 0.475

Using formula IV (a)

$$\cos y = \frac{2.01 - 0.475}{2} = 0.767 \text{ and hence } y = 40°.$$

Using formula IV (b)

$$\sinh x = 0.472/\sin 40° = 0.735.$$

With the indicator on this value on C, turn the rule over, bring indices together, and read $x = 0.68$ on Sh. Hence $z = 0.68 + j\,40°$. Compare with Example (b), page 16.

Finding z for tanh z = ρ/θ.

If ρ and θ are known and $z = x + jy = \text{arc tanh}\, \rho \angle\theta$ is to be found, first find $u = \rho \cos\theta$ and $v = \rho \sin\theta$. Then $\tanh(x + jy) = u + jv$. It can readily be shown by use of the relation

(21) $$\tanh(z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$$

that $\tanh(x - jy) = u - jv$, and hence

(22) $$x + jy = \text{arctanh}\,(u + jv)$$
$$x - jy = \text{arctanh}\,(u - jv).$$

Adding,

$$2x = \text{arctanh}\,(u + jv) + \text{arctanh}\,(u - jv);$$

and by using equation (21)

$$\tanh 2x = \frac{u + jv + u - jv}{1 + (u + jv)(u - jv)}, \text{ or}$$

V (a) $$\tanh 2x = \frac{2u}{1 + u^2 + v^2} = \frac{2u}{1 + \rho^2}$$

Subtracting the equations (22),

$$2\,jy = \text{arctanh}\,(u + jv) - \text{arctanh}\,(u + jv), \text{ and hence}$$

$$\tanh 2jy = \frac{u + jv - (u - jv)}{1 - (u + jv)(u - jv)} = \frac{2jv}{1 - (u^2 + v^2)}$$

Finally, $\tanh 2jy = j \tan 2y$, and consequently

V(b) $$\tan 2y = \frac{2v}{1 - \rho^2}$$

Therefore x and y may be found from formulas V(a) and V(b), respectively. Expressed explicitly, they are

$$x = \tfrac{1}{2}\,\text{arctanh}\, 2u/(1 + \rho^2);\; y = \tfrac{1}{2}\,\text{arctan}\, 2v/(1 - \rho^2).$$

The values of z for $\coth z = \rho\angle\theta$, $\text{sech}\, z = \rho\angle\theta$, and $\text{csch}\, z = \rho\angle\theta$ may be found, if desired, by using the reciprocal relationships; that is, by finding z for $\tanh z = (1/\rho)\angle -\theta$, $\cosh z = (1/\rho)\angle -\theta$, and $\sinh z = (1/\rho)\angle -\theta$.

ILLUSTRATIVE APPLIED PROBLEM

Suppose the propagation constant $\gamma = \alpha + j\beta$ of a line is to be found by the formula

$$\tanh \gamma l = \sqrt{\frac{\mathbf{Z}_{ss}}{\mathbf{Z}_{so}}},$$

where $\mathbf{Z}_{ss} = 3.520 \underline{/-76.3^\circ}$ ohms, and $\mathbf{Z}_{so} = 1430 \underline{/72.7^\circ}$ ohms, and $l = 30$ miles. Then

$$\tanh 30\gamma = \sqrt{\frac{3520 \underline{/-76.3}}{1430 \underline{/72.7^\circ}}}$$

$$= 1.568 \underline{/-79.5^\circ} = \rho \underline{/\theta}.$$

To calculate this, set 143 of C opposite 352 on D and read 1.568 on the upper $\sqrt{}$ scale at the C index. Then calculate $(-76.3 - 72.7)/2 = -79.5$. Move the left C index to 1.568 on D, move indicator to 79.5 on S (read right to left), and find $u = 0.2855$. Move right C- index to 1.568 on D, indicator to 79.5° on S, and read 1.54, on D. Thus, since θ is in the fourth quadrant, $v = -1.54$ and $\tanh 30\gamma = 0.2855 - j\,1.54 = u + jv$

Now by (Va), $\tanh 2x = \dfrac{2\,(0.2855)}{1 + 1.568^2} = \dfrac{0.575}{1 + 2.46} = \dfrac{0.575}{3.46} = 0.166$

Set hairline on 0.166 on C, read $2x = 0.1675$ on Th, from which $x = 0.0838$. From formula (Vb)

$$\tan 2y = \frac{2\,(-1.54)}{1 - 2.46} = \frac{-3.08}{-1.46} = 2.11$$

Set 2.11 on CI and read 64.2° on T (read right to left). Thus $2y = 64.2^\circ$, or $64.2 + 180^\circ = 244.2^\circ$; then $y = 32.1^\circ$ or 122.1°. Suppose that other known conditions about the line suffice to determine that $y = 122.1^\circ$ is the proper value. Then $30\gamma = 0.0838 + j\,122.1^\circ$; or, since $122.1^\circ = 2.135$ radians, $30\gamma = 0.0838 + j\,2.135$, whence $\gamma = 0.00279 + j\,0.0712$. To change this to polar form, set right C index on 712 of D, move hairline to 279 of D, and note that $0.0712/.00279^* = 25$ approximately. Hence read $\varphi = 2.25^\circ$ on ST, and find $\theta = 90 - \varphi = 87.75^\circ$. The value of this ρ is nearly 0.0712, or about 0.0713. Hence

$$\gamma = 0.0713 \underline{/87.75^\circ}.$$

PART 5. CIRCULAR FUNCTIONS OF COMPLEX ARGUMENTS

By means of the formulas on page 9 the following relations may be proved:

$\sin (x + jy) = \sin x \cosh y + j \cos x \sinh y,$
$\cos (x + jy) = \cos x \cosh y - j \sin x \sinh y.$

The analogies between these results and those for $\sin (x + y)$ and $\cos (x + y)$ in trigonometry should be carefully noted. Observe also that here the x is associated with a *circular* function and the y is associated with a *hyperbolic* function.

These formulas express the sine and cosine of a complex argument as a complex number in terms of its components.

If $\sin (x + jy)$ is to be expressed as a complex number in polar form, reasoning similar to that in Part 3 leads to the analogous results:

VI (a) $$\tan \theta = \frac{\tanh y}{\tan x},$$

(b) $$\rho = \frac{\sin x \cosh y}{\cos \theta}.$$

Moreover, $\rho = \sqrt{\sin^2 x + \sinh^2 y}$, and this formula may be used as a check or as the basis of an alternative method of computing ρ from the expression $\rho \underline{/\theta_a} = \sin x + j \sinh y$.

Similarly, if $\cos (x + jy)$ is to be expressed as a complex number in polar form, first find ρ and θ for the conjugate variable $x - jy$ in $\cos (x - jy)$ by the following formulas:

VII (a) $$\tan \theta = \tan x \tanh y,$$

(b) $$\rho = \frac{\sin x \sinh y}{\sin \theta}$$

Then $\cos (x + jy) = \rho \underline{/-\theta}$.

Finally, $\tan (x + jy)$ may be readily found by finding the sine and cosine in polar form and taking the quotient.

The close analogy between these formulas for circular functions and the corresponding ones for hyperbolic functions, enables the computer to use similar methods of calculation. In the case of *hyperbolic* functions, the x of $x + jy$ is associated with a hyperbolic function. Symbolically, x is preceded by an h, as in $\sinh x$. In the case of the *circular* functions, the y of $x + jy$ is associated with a hyperbolic function, or symbolically, is preceded by an h, as in $\sinh y$. An adjustment of algebraic sign is called for in the case of the cosine.

EXAMPLES:

(a) Find $\sin (30^\circ + j\,0.48)$. With C and D indices together, set indicator on 0.48 of Th. Move slide until 30° on T is under hairline. Note $\tanh 0.48/\tan 30 = 0.774$, approximately. Move indicator to C-index, and then bring indices together, reading $\theta = 37.7^\circ$ on T. Or, move indicator to 0.774 on C and read θ on T.

Set indicator on 0.48 of Sh; move slide until 0.48 on Th is under hairline; move indicator to sin 30° on S; move slide until $\theta = 37.7^\circ$ on S (reading right to left) is under indicator. Read $\rho = 0.707$ on D at right C-index. Then $\sin (30^\circ + j\,0.48) = 0.707 \underline{/37.7^\circ}$.

(b) Find $\cos (30^\circ + j\,0.48)$. With C and D indices together, set indicator on 0.48 of Th; move right index of slide under hairline, then move hairline to 30° on T. Bring indices together, and read $\theta = 14.45^\circ$ on T.

Set indicator on 0.48 of Sh; move slide until 14.45° on S is under hairline; move indicator to 30° on S; read $\rho = 1$ on D. Then $\cos (30^\circ + j\,0.48) = 1 \underline{/-14.45^\circ}$.

(c) Find $\tan (30^\circ + j\,0.48)$. From examples (a) and (b) above:

$$\tan (30^\circ + j\,0.48) = \frac{\sin (30^\circ + j\,0.48)}{\cos (30^\circ + j\,0.48)} = \frac{0.707 \underline{/37.7^\circ}}{1 \underline{/-14.45^\circ}} = 0.707 \underline{/52.15^\circ}$$

If $z = x + jy$ is to be found so that $\sin z = u + jv$ or $\cos z = u + jv$, methods similar to those outlined in Part 4 yield the following formulas from which x and y may be found.

VIII (a) $$\sin x = \frac{U_c - V_c}{2}, \text{ where}$$

$$U_c = \sqrt{(u+1)^2 + v^2} \text{ and } V_c = \sqrt{(u-1)^2 + v^2}$$

(b) $$\sinh y = v/\cos x$$

IX (a) $$\cos x = \frac{U_c - V_c}{2}, \text{ where } U_c \text{ and } V_c \text{ are defined as above.}$$

(b) $$\sinh y = v/\sin x$$

X (a) $$\tan 2x = \frac{2u}{1 - \rho^2}$$

(b) $$\tanh 2y = \frac{2v}{1 + \rho^2}$$

Examples:

(a) Find $z = x + jy$ if $\sin z = 0.707\,\underline{/37.7^\circ}$
Here $u = 0.558$; $v = 0.431$

$$\sin x = \left\{\sqrt{1.558^2 + 0.431^2} - \sqrt{0.442^2 + 0.431^2}\right\}/2$$

$$= (1.617 - 0.617)/2 = 0.500$$

Then $x = 30^\circ$, and $\sinh y = 0.431/\cos 30^\circ$, $y = 0.48$
Thus $x + jy = 30^\circ + j\,0.48$

(b) Find $z = x + jy$ if $\cos(x + jy) = 1\,/{-14.45^\circ}$
Here $u = 0.967$; $v = -\,0.2495$

$$\cos x = \left\{\sqrt{(1.967)^2 + (-\,0.2495)^2} - \sqrt{0.033^2 + (-\,0.2495)^2}\right\}/2$$

$$= (1.984 - 0.251)\,/2 = 0.866 = \cos 30^\circ$$

$$\sinh y = \frac{-(-\,0.2495)}{\sin 30^\circ} = 0.499$$

Then $x = 30^\circ$, $y = 0.480$, and $z = 30^\circ + j\,0.48$

(c) Find $x + jy$ if $\tan(x + jy) = 0.707\,\underline{/52.15^\circ}$
First find $u = 0.707 \cos 52.15^\circ = 0.433$, and
$v = 0.707 \sin 52.15^\circ = 0.557$.
Then $\tan(x + jy) = 0.433 + j\,0.557$, and $\rho^2 = 0.5$

$$\tan 2x = \frac{2\,(0.433)}{1 - 0.5} = 1.732$$

Hence $2x = 60^\circ$ and $x = 30^\circ$. Continuing,

$$\tanh 2y = \frac{2\,(0.557)}{1 + 0.5} = 0.743, \text{ which yields } 2y = 0.96 \text{ on Th,}$$

and hence $y = 0.48$. Finally, $x + jy = 30^\circ + j\,0.48$ or $0.524 + j\,0.48$, if 30° is changed to radians.

FOR DOUBLE T SCALE (MODEL 4)

(i) to x of the 2 no.(x, y) on D set on index of the slide. Set indicator over the value of y on D and read θ on T scale. If y < x, θ < 45°, ξ is found on the upper T scale. If y > x, θ > 45° ξ is found on lower T scale.

(ii) Move the slide until θ on the S scale is under the hairline. Interchange the indices of C if necessary. Read ρ on D under C index.

ALTERNATE METHOD:

(i) To y of the 2 no. (x, y) on DI set an index of slide. Set indicator over the value of x on DI and read θ on T scale. If $y < x$, $\theta < 45°$, ε is read on upper T scale. If $y > x$, $\theta > 45°$, ε is read on lower T scale.

(ii) Move indicator over θ on S scale. Interchange indices of C if necessary. Read ρ on DI under hairline.

$$X + iY = \rho \angle \theta°$$

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