how to use LOG LOG SLIDE RULES

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PICKETT
THE WORLD'S MOST ACCURATE SLIDE RULES

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PREFACE

A computer who must make many difficult calculations usually has a book of tables of the elementary mathematical functions, or a slide rule, close at hand. In many cases the slide rule is a very convenient substitute for a book of tables. It is, however, much more than that, because by means of a few simple adjustments the actual calculations can be carried through and the result obtained. One has only to learn to read the scales, how to move the slide and indicator, and how to set them accurately, in order to be able to perform long and otherwise difficult calculations.

When people have difficulty in learning to use a slide rule, usually it is not because the instrument is difficult to use. The reason is likely to be that they do not understand the mathematics on which the instrument is based, or the formulas they are trying to evaluate. Some slide rule manuals contain relatively extensive explanations of the theory underlying the operations. Some explain in detail how to solve many different types of problems—for example, various cases which arise in solving triangles by trigonometry. In this manual such theory has deliberately been kept to a minimum. It is assumed that the *theory* of exponents, of logarithms of trigonometry, and of the slide rule is known to the reader, or will be recalled or studied by reference to formal textbooks on these subjects. This is a brief manual on operational technique and is not intended to be a textbook or workbook. Relatively few exercises are included, and the answers of these (to slide rule accuracy) are given immediately so that learning may proceed rapidly, by self-correction if necessary. Any textbook on mathematics which contains problems suitable for slide rule calculation, and their answers, will provide additional practice.

Some of the special scales described in this manual may not be available on your slide rule. All of the illustrations and problems shown can be worked on the slide rule you purchased. However, the special scales simplify the calculations. Pickett Slide Rules are available with all of the special scales shown in this manual.

Form MLL-1

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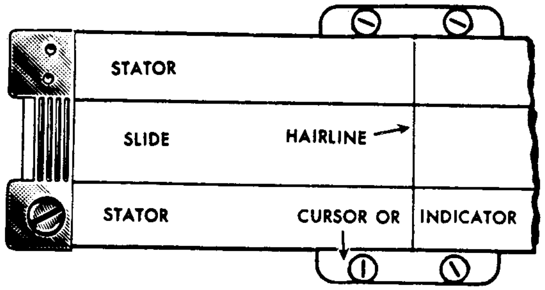
PART 1. SLIDE RULE OPERATION

INTRODUCTION

The slide rule is a fairly simple tool by means of which answers to involved mathematical problems can be easily obtained. To solve problems easily and with confidence it is necessary to have a clear understanding of the operation of your slide rule. Speed and accuracy will soon reward the user who makes a careful study of the scale arrangements and the manual.

The slide rule consists of three parts: (1) the stator (upper and lower bars); (2) the slide; (3) the cursor or indicator. The scales on the bars and slide are arranged to work together in solving problems. The hairline on the indicator is used to help the eyes in reading the scales and in adjusting the slide.

Fig. 1



Each scale is named by a letter (A, B, C, D, L, S, T) or other symbol at the left end.

The table below shows some of the mathematical operations which can be done easily and quickly with an ordinary slide rule.

OPERATIONS	INVERSE OPERATIONS
Multiplying two or more numbers	Dividing one number by another
Squaring a number	Finding the square root of a number
Cubing a number	Finding the cube root of a number
Finding the logarithm of a number	Finding a number whose logarithm is known
Finding the sine, cosine, or tangent of an angle	Finding an angle whose sine, cosine, or tangent is known

Various *combinations* of these operations (such as multiplying two numbers and then finding the square root of the result) are also easily done. Numbers can be added or subtracted with an ordinary slide rule, but it is usually easier to do these operations by arithmetic.*

In order to use a slide rule, a computer must know: (1) how to read the scales; (2) how to "set" the slide and indicator for each operation to be done; and (3) how to determine the decimal point in the result.

*By putting special scales on a slide rule, these and certain other operations much more difficult than those shown in the table above can be done easily.

HOW TO READ THE SCALES

The scale labeled C (on the slide) and the scale D (on the stator bar) are used most frequently. These two scales are exactly alike. The total length of these scales has been separated into many smaller parts by fine lines called "graduations."

Some of these lines on the D scale have large numerals (1, 2, 3, etc.) printed just below them. These lines are called *primary* graduations. On the C scale the numerals are printed above the corresponding graduations. A line labeled 1 at the left end is called the *left* index. A line labeled 1 at the right end is called the *right* index.

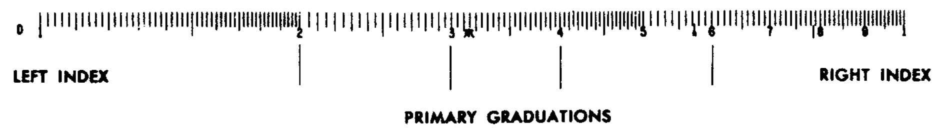


Fig. 2

Next notice that the distance between 1 and 2 on the D scale has been separated into 10 parts by shorter graduation lines. These are the *secondary* graduations. (On 10 inch slide rules these lines are labeled with smaller numerals 1, 2, 3, etc. On 6 inch rules these lines are not labeled.) Each of the spaces between the larger numerals 2 and 3, between 3 and 4, and between the other primary graduations is also sub-divided into 10 parts. Numerals are not printed beside these smaller secondary graduations because it would crowd the numerals too much.

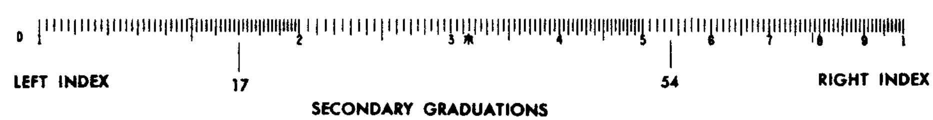


Fig. 3.

When a number is to be located on the D scale, the *first* digit is located by use of the *primary* graduations. The *second* digit is located by use of the *secondary* graduations. Thus when the number 17 is located, the 1 at the left index represents the 1 in 17. The 7th secondary graduation represents the 7. When 54 is to be located, look first for primary graduation 5, and then for secondary graduation 4 in the space immediately to the right.

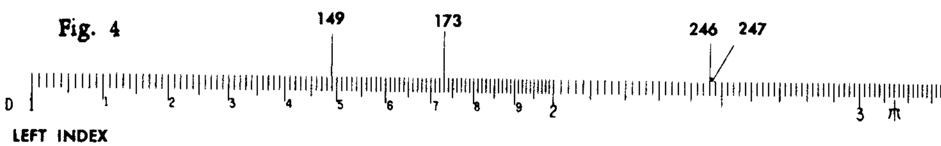
There are further sub-divisions, or *tertiary graduations*, on all slide rules. The meaning of these graduations is slightly different at different parts of the scale. It is also different on a 6 inch slide rule than on a 10 inch rule. For this reason a separate explanation must be given for each.

Tertiary graduations on 10 inch rules.

The space between each secondary graduation at the left end of the rule (over to primary graduation **2**) is separated into ten parts, but these shortest graduation marks are not numbered. In the middle part of the rule, between the primary graduations **2** and **4**, the smaller spaces between the *secondary* graduations are separated into five parts. Finally, the still smaller spaces between the secondary graduations at the right of **4** are separated into only two parts.

To find 173 on the D scale, look for primary division **1** (the left index), then for secondary division 7 (numbered) then for smaller subdivision 3 (not numbered, but found as the 3rd very short graduation to the right of the longer graduation for 7).

Fig. 4



Similarly, 149 is found as the 9th small graduation mark to the right of the 4th secondary graduation mark to the right of primary graduation **1**.

To find 246, look for primary graduation **2**, then for the 4th secondary graduation after it (the 4th long line), then for the 3rd small graduation after it. The smallest spaces in this part of the scale are fifths. Since $\frac{3}{5} = \frac{6}{10}$, then the third graduation, marking *three fifths*, is at the same point as *six tenths* would be.

Tertiary graduations on 6 inch rules.

The space between each secondary graduation at the left end of the rule (over to primary graduation 2) is separated into five parts. In the middle of the rule, between the primary graduations 2 and 5, the smaller spaces between the secondary graduations are separated into two parts. Finally, the still smaller spaces between the secondary graduations at the right of 5 are not subdivided.

To find 170 on the D scale, look for the primary division 1 (the left index), then for the secondary division 7 which is the 7th secondary graduation.

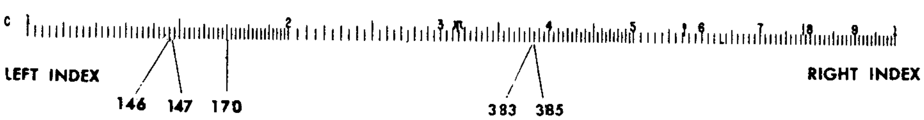


Fig. 5

To find 146, look for the primary graduation 1, then for the 4th secondary graduation after it (the 4th secondary line), then for the 3rd small graduation after it. The smallest spaces in this part of the scale are fifths. Since 3/5 = 6/10, then the third graduation, marking *three fifths*, is at the same point as *six tenths* would be.

The number 147 would be half of a small space beyond 146. With the aid of the hairline on the runner the position of this number can be located approximately by the eye. The small space is mentally "split" in half.

The number 385 is found by locating primary graduation 3 and then secondary graduation 8 (the 8th long graduation after 3). Following this, one observes that between secondary graduations 8 and 9 there is one short mark. Think of this as the "5 tenths" mark, which represents 385. The location of 383 can be found approximately by mentally "splitting" the space between 380 and 385 into fifths, and estimating where the 3rd "fifths" mark would be placed. It would be just a little to the right of halfway between 380 and 385.

On the scale below are some sample readings.

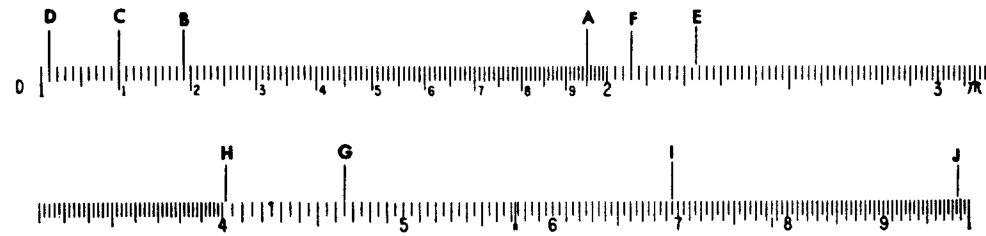


Fig. 6

A: 195	F: 206
B: 119	G: 465
C: 110	H: 402
D: 101	I: 694
E: 223	J: 987

The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, used in writing numbers are called *digits*. One way to describe a number is to tell how many digits are used in writing it. Thus 54 is a "two-digit number", and 1,348,256 is a "seven-digit number." In many computations only the first two or three digits of a number need to be used to get an approximate result which is accurate enough for practical purposes. Usually not more than the first three digits of a number can be "set" on a six inch slide rule scale. In many practical problems this degree of accuracy is sufficient. When greater accuracy is desired, a ten inch slide rule is generally used.

MULTIPLICATION

Numbers that are to be multiplied are called factors. The result is called the product. Thus, in the statement $6 \times 7 = 42$, the numbers 6 and 7 are factors, and 42 is the product.

EXAMPLE: Multiply 2×3

Setting the Scales: Set the left index of the C scale on 2 of the D scale. Find 3 on the C scale, and below it read the product, 6 on the D scale.

Think: The length for 2 plus the length for 3 will be the length for the product. This length, measured by the D scale, is 6.

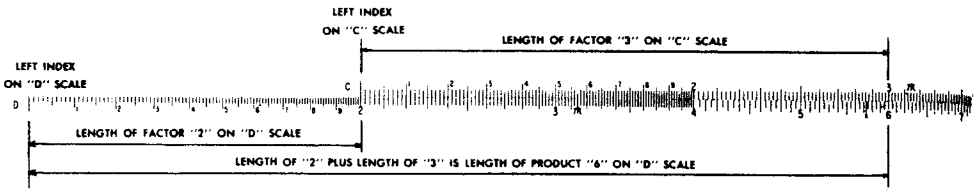


Fig. 7

EXAMPLE: Multiply 4×2

Setting the Scales: Set the left index of the C scale on 4 of the D scale. Find 2 on the C scale, and below it read the product, 8, on the D scale.

Think: The length for 4 plus the length for 2 will be the length for the product. This length, measured by the D scale, is 8.

***Rule for Multiplication:* Over one of the factors on the D scale, set the index of the C scale. Locate the other factor on the C scale, and directly below it read the product on the D scale.**

EXAMPLE. Multiply 2.34×36.8

Estimate the result: First note that the result will be roughly the same as 2×40, or 80; that is, there will be two digits to the left of the decimal point. Hence, we can ignore the decimal points for the present and multiply as though the problem was 234×368.

Set the Scales: Set the left index of the C scale on 234 of the D scale. Find 368 on the C scale and read product 861 on the D scale.

Think: The length for 234 plus the length for 368 will be the length for the product. This length is measured on the D scale. Since we already knew the result was somewhere near 80, the product must be 86.1, approximately.

EXAMPLE: Multiply 28.3×5.46

Note first that the result will be about the same as 30×5, or 150. Note also that if the left index of the C scale is set over 283 on the D scale, and 546 is then found on the C scale, the slide projects so far to the right of the rule that the D scale is no longer below the 546. When this happens, the other index of the C scale must be used. That is, set the *right* index on the C scale over 283 on the D scale. Find 546 on the C scale and below it read the product on the D scale. The product is 154.5.

These examples illustrate how in simple problems the decimal point can be placed by use of an estimate.

PROBLEMS	ANSWERS
1. 15×3.7	55.5
2. 280×0.34	95.2
3. 753×89.1	67,100
4. 9.54×16.7	159.3
5. 0.0215×3.79	0.0815

DIVISION

In mathematics, division is the opposite or *inverse* operation of multiplication. In using a slide rule this means that the process for multiplication is reversed. To help in understanding this statement, set the rule to multiply 2×4 (see page 9). Notice the result 8 is found on the D scale under 4 of the C scale. Now to divide 8 by 4 these steps are reversed. First find 8 on the D scale, set 4 on the C scale over it, and read the result 2 on the D scale under the index of the C scale.

Think: From the length for 8 (on the D scale) *subtract* the length for 4 (on the C scale). The length for the difference, read on the D scale, is the result, or quotient.

With this same setting you can read the quotient of $6 \div 3$, or $9 \div 4.5$, and in fact all divisions of one number by another in which the result is 2.

***Rule for Division:* Set the *divisor* (on the C scale) opposite the number to be divided (on the D scale). Read the result, or quotient, on the D scale under the index of the C scale.**

EXAMPLES:

(a) Find $63.4 \div 3.29$. The quotient must be near 20, since $60 \div 3 = 20$. Set indicator on 63.4 of the D scale. Move the slide until 3.29 of the C scale is under the hairline. Read the result 19.27 on the D scale at the C index.

(b) Find $26.4 \div 47.7$. Since 26.4 is near 25, and 47.7 is near 50, the quotient must be roughly $25/50 = ½ = 0.5$. Set 47.7 of C opposite 26.4 of D, using the indicator to aid the eyes. Read 0.553 on the D scale at the C index.

PROBLEMS	ANSWERS
1. $83 \div 7$	11.86
2. $75 \div 92$	0.815
3. $137 \div 513$	0.267
4. $17.3 \div 231$	0.0749
5. $8570 \div .0219$	391,000

DECIMAL POINT LOCATION

In the discussion which follows, it will occasionally be necessary to refer to the number of "digits" and number of "zeros" in some given numbers.

When numbers are greater than 1 the number of *digits* to the left of the decimal point will be counted. Thus 734.05 will be said to have 3 digits. Although as written the number indicates accuracy to *five* digits, only three of these are at the left of the decimal point.

Numbers that are less than 1 may be written as *decimal fractions*.* Thus .673, or six-hundred-seventy-three thousandths, is a decimal fraction. Another example is .000465. In this number three zeros are written to show where the decimal point is located. One way to describe such a number is to tell how many zeros are written to the right of the decimal point before the first non-zero digit occurs.

In scientific work a zero is often written to the left of the decimal point, as in 0.00541. This shows that the number in the units' place is definitely 0, and that no digits have been carelessly omitted in writing or printing. The zeros will *not* be counted unless they are (a) at the *right* of the decimal point, (b) before or at the *left* of the first non-zero digit, and (c) are not between other digits. The number 0.000408 will be said to have 3 zeros (that is, the number of zeros between the decimal point and the 4).

In many, perhaps a majority, of the problems met in genuine applications of mathematics to practical affairs, the position of the decimal point in the result can be determined by what is sometimes called "common sense." There is usually only one place for the decimal point in which the answer is "reasonable" for the problem. Thus, if the calculated speed in miles per hour of a powerful new airplane comes out to be 4833, the decimal point clearly belongs between the 3's, since 48 m.p.h. is too small, and 4833 m.p.h. is too large for such a plane. In some cases, however, the data are such that the position of the point in the final result is not easy to get by inspection.

Another commonly used method of locating the decimal point is by estimation or approximation. For example, when the slide rule is used to find 133.4×12.4, the scale reading for the result is 1655, and the decimal point is to be determined. By rounding off the factors to 133.0×10.0, one obtains 1330 by mental arithmetic. The result would be somewhat greater than this but certainly contains four digits on the left of the decimal point. The answer, therefore, must be 1655.

In scientific work numbers are often expressed in *standard form*. For example, 428 can be written 4.28×10^2, and 0.00395 can be written as 3.95×10^{-3}. When a number is written in standard form it always has two factors. The first factor has one digit (not a zero) on the left of the decimal point, and usually other digits on the right of the decimal point. The other factor is a power of 10 which places the decimal point in its true position if the indicated multiplication is carried out. In many types of problems this method of writing numbers simplifies the calculation and the location of the decimal point.

*Only positive real numbers are being considered in this discussion.

When a number is written in standard form, the exponent of 10 may be called "the characteristic." It is the characteristic of the logarithm of the number to base 10. The characteristic may be either a positive or a negative number. Although the rule below appears long, in actual practice it may be used with great ease.

***Rule.* To express a number in standard form:**
(a) place a decimal point at the right of the first non-zero digit.*
(b) start at the right of the first non-zero digit in the original number and count the digits and zeros passed over in reaching the decimal point. The result of the count is the numerical value of the characteristic, or exponent of 10. If the original decimal point is toward the right, the characteristic is *positive* (+). If the original decimal point is toward the left, the characteristic is *negative* (−). Indicate that the result of (a) is to be multiplied by 10 with the exponent thus determined in (b).

EXAMPLES:

Number	Number in standard form.	Characteristic
(a) 5,790,000	5.79×10^6	6
(b) 0.000283	2.83×10^{-4}	−4
(c) 44	4.4×10^1	1
(d) 0.623	6.23×10^{-1}	−1
(e) 8.15	8.15×10^0	0
(f) 461,328	4.61328×10^5	5
(g) 0.0000005371	5.371×10^{-7}	−7
(h) 0.0306	3.06×10^{-2}	−2
(i) 80.07	8.007×10^1	1

If a number given in standard form is to be written in "ordinary" form, the digits should be copied, and then starting at the right of the first digit the number of places indicated by the exponent should be counted, supplying zeros as necessary, and the point put down. If the exponent is positive, the count is toward the right; if negative, the count is toward the left. This converse application of the rule may be verified by studying the examples given above.

Consider now the calculation of $5{,}790{,}000 \times 0.000283$. From examples (a) and (b) above, this can be written $5.79 \times 10^6 \times 2.83 \times 10^{-4}$, or by changing order and combining the exponents of 10, as $5.79 \times 2.83 \times 10^2$. Then since 5.79 is near 6, and 2.83 is near 3, the product of these two factors is known to be near 18. The multiplication by use of the C and D scales shows it to be about 16.39, or 1.639×10^1. Hence, $5.79 \times 2.83 \times 10^2 = 1.639 \times 10^1 \times 10^2 = 1.639 \times 10^3 = 1639$. If, however, one has

$5{,}790{,}000 \div 0.000283$, the use of standard form yields

$$\frac{5.79 \times 10^6}{2.83 \times 10^{-4}} = 2.04 \times 10^{6-(-4)} = 2.04 \times 10.^{10}$$

In scientific work the result would be left in this form, but for popular consumption it would be written as 20,400,000,000. The general rule is as follows.

*In using this rule, "first" is to be counted from the left; thus, in 3246, the digit 3 is "first."

Rule. To determine the decimal point, first express the numbers in standard form. Carry out the indicated operations of multiplication or division, using the laws of exponents* to combine the exponents until a single power of 10 is indicated. If desired, write out the resulting number, using the final exponent of 10 to determine how far, and in what direction, the decimal point in the coefficient should be moved.

CONTINUED PRODUCTS

Sometimes the product of three or more numbers must be found. These "continued" products are easy to get on the slide rule.

EXAMPLE: Multiply $38.2 \times 1.65 \times 8.9$.

Estimate the result as follows: $40 \times 1 \times 10 = 400$. The result should be, very roughly, 400.

Setting the Scales: Set left index of the C scale over 382 on the D scale. Find 165 on the C scale, and *set the hairline* on the indicator on it.** Move the index on the slide under the hairline. In this example if the *left* index is placed under the hairline, then 89 on the C scale falls outside the D scale. Therefore move the *right* index under the hairline. Move the hairline to 89 on the C scale and read the result (561) under it on the D scale.

Below is a *general rule* for continued products: $a \times b \times c \times d \times e \cdots$

Set hairline of indicator at a on D scale.
Move index of C scale under hairline.
Move hairline over b on the C scale.
Move index of C scale under hairline.
Move hairline over c on the C scale.
Move index of C scale under hairline.

Continue moving hairline and index alternately until all numbers have been set.

Read result under the hairline on the D scale.

PROBLEMS	ANSWERS
1. $2.9 \times 3.4 \times 7.5$	73.9
2. $17.3 \times 43 \times 9.2$	6,840
3. $343 \times 91.5 \times 0.00532$	167
4. $19 \times 407 \times 0.0021$	16.24
5. $13.5 \times 709 \times 0.567 \times 0.97$	5260

COMBINED MULTIPLICATION AND DIVISION

Many problems call for both multiplication and division.

EXAMPLE: $\dfrac{42 \times 37}{65}$.

*See any textbook on elementary algebra. The theory of exponents and the rules of operation with signed numbers are both involved in a complete treatment of this topic. In this manual it is assumed that the reader is familiar with this theory.

**The product of 382×165 could now be read under the hairline on the D scale, but this is not necessary.

First, set the division of 42 by 65; that is, set 65 on the C scale opposite 42 on the D scale.* Move the hairline on indicator to 37 on the C scale. Read the result 239 on the D scale under the hairline. Since the fraction $\frac{42}{65}$ is about equal to $\frac{2}{3}$, the result is about two-thirds of 37, or 23.9.

EXAMPLE: $\dfrac{273 \times 548}{692 \times 344}$

Set 692 on the C scale opposite 273 on the D scale. Move the hairline to 548 on the C scale. Move the slide to set 344 on the C scale under the hairline. Read the result .628 on the D scale under the C index.

In general, to do computations of the type $\dfrac{a \times c \times e \times g \cdots}{b \times d \times f \times h \cdots}$, set the rule to divide the first factor in the numerator a by the first factor in the denominator b, move the hairline to the next factor in the numerator c; move the slide to set next factor in denominator, d, under the hairline. Continue moving hairline and slide alternately for other factors (e, f, g, h, *etc.*). Read the result on the D scale. If there is one more factor in the numerator than in the denominator, the result is under the hairline. If the number of factors in numerator and denominator is the same, the result is under the C index. Sometimes the slide must be moved so that one index replaces the other.**

EXAMPLE: $\dfrac{2.2 \times 2.4}{8.4}$

If the rule is set to divide 2.2 by 8.4, the hairline cannot be set over 2.4 of the C scale and at the same time remain on the rule. Therefore the hairline is moved to the C index (opposite 262 on the D scale) and the slide is moved end for end to the right (so that the *left* index falls under the hairline and over 262 on the D scale). Then the hairline is moved to 2.4 on the C scale and the result .63 is read on the D scale.

If the number of factors in the numerator exceeds the number in the denominator by more than one, the numbers may be grouped, as shown below. After the value of the *group* is worked out, it may be multiplied by the other factors in the usual manner.

$$\left(\frac{a \times b \times c}{m \times n}\right) \times d \times e$$

PROBLEMS	ANSWERS
1. $\dfrac{27 \times 43}{19}$	61.1
2. $\dfrac{5.17 \times 1.25 \times 9.33}{4.3 \times 6.77}$	2.07
3. $\dfrac{842 \times 2.41 \times 173}{567 \times 11.52}$	53.7
4. $\dfrac{0.0237 \times 3970 \times 32 \times 6.28}{0.00029 \times 186000}$	351

*The quotient, .646, need not be read.

**This statement assumes that up to this point only the C and D scales are being used. Later sections will describe how this operation may be avoided by the use of other scales.

PROPORTION

Problems in proportion are very easy to solve. First notice that when the index of the C scale is opposite 2 on the D scale, the ratio 1 : 2 or $\frac{1}{2}$ is at the *same time* set for all other opposite graduations; that is, 2 : 4, or 3 : 6, or 2.5 : 5, or 3.2 : 6.4, etc. It is true in general that for any setting the numbers for *all pairs of opposite graduations have the same ratio.* Suppose one of the terms of a proportion is unknown. The proportion can be written as $\frac{a}{b} = \frac{c}{x}$, where a, b, and c, are known numbers and x is to be found.

***Rule:* Set a on the C scale opposite b on the D scale. Under c on the C scale read x on the D scale.**

EXAMPLE: Find x if $\frac{3}{4} = \frac{5}{x}$.

Set 3 on C opposite 4 on D. Under 5 on C read 6.67 on D.

The proportion above could also be written $\frac{b}{a} = \frac{x}{c}$, or "inverted," and exactly the same rule could be used. Moreover, if C and D are interchanged in the above rule, it will still hold if "under" is replaced by "over." It then reads as follows:

Set a on the D scale opposite b on the C scale. Over c on the D scale read x on the C scale.

***Rule:* In solving proportions, keep in mind that the position of the numbers as set on the scales is the same as it is in the proportion written in the form $\frac{a}{b} = \frac{c}{d}$**

Proportions can also be solved *algebraically*. Then $\frac{a}{b} = \frac{c}{x}$ becomes $x = \frac{bc}{a}$, and this may be computed as combined multiplication and division.

PROBLEMS	ANSWERS
1. $\frac{x}{42.5} = \frac{13.2}{1.87}$	300.
2. $\frac{90.5}{x} = \frac{3.42}{1.54}$	40.7
3. $\frac{43.6}{89.2} = \frac{x}{2550}$	1247.
4. $\frac{0.063}{0.51} = \frac{34.1}{x}$	276.
5. $\frac{18}{91} = \frac{13}{x}$	65.7

PART 2. USE OF CERTAIN SPECIAL SCALES

THE CI AND DI SCALES

It should be understood that the use of the CI and DI scales does not increase the power of the instrument to solve problems. In the hands of an experienced computer, however, these scales are used to reduce the number of settings or to avoid the awkwardness of certain settings. In this way the speed can be increased and errors minimized.

The CI scale on the slide is a C scale which *increases from right to left.* It may be used for finding reciprocals. When any number is set under the hairline on the C scale its reciprocal is found under the hairline on the CI scale, and conversely.

EXAMPLES:

(a) Find 1/2.4. Set 2.4 on C. Read .417 directly above on CI.

(b) Find 1/60.5. Set 60.5 on C. Read .0165 directly above on CI. Or, set 60.5 on CI., read .0165 directly below on C.

The CI scale is useful in replacing a division by a multiplication. Since $\frac{a}{b} = a \times 1/b$, any division may be done by multiplying the numerator (or dividend) by the reciprocal of the denominator (or divisor). This process may often be used to avoid settings in which the slide projects far outside the rule.

EXAMPLES:

(a) Find 13.6 ÷ 87.5. Consider this as 13.6 × 1/87.5. Set left index of the C scale on 13.6 of the D scale. Move hairline to 87.5 on the CI scale. Read the result, .155, on the D scale.

(b) Find 72.4 ÷ 1.15. Consider this as 72.4 × 1/1.15. Set right index of the C scale on 72.4 of the D scale. Move hairline to 1.15 on the CI scale. Read 63.0 under the hairline on the D scale.

An important use of the CI scale occurs in problems of the following type.

EXAMPLE: Find $\frac{13.6}{4.13 \times 2.79}$ This is the same as $\frac{13.6 \times (1/2.79)}{4.13}$.

Set 4.13 on the C scale opposite 13.6 on the D scale. Move hairline to 2.79 on the CI scale, and read the result, 1.180, on the D scale.

By use of the CI scale, factors may be transferred from the numerator to the denominator of a fraction (or vice-versa) in order to make the settings more readily. Also, it is sometimes easier to get $a \times b$ by setting the hairline on a, pulling b on the CI scale under the hairline, and reading the result on the D scale under the index.

The DI scale (inverted D scale) below the D scale corresponds to the CI scale on the slide. Thus the D and DI scales together represent reciprocals. The DI scale has several important uses, of which the following is representative.

Expressions of the type $1/X$, where X is some complicated expression or formula, may be computed by first finding the value of X. If the result for X falls on D, then $1/X$ may be read under the hairline on DI.

EXAMPLE:

(a) Find $\frac{1}{0.265 \times 138}$. Multiply 0.265×138 using the C and D scales. Read the reciprocal .0273 under the hairline on the DI scale. Or set the hairline on 265 of the DI scale, pull 138 of the C scale under the hairline, and read the result on the D scale under the left index of the C scale. This is equivalent to writing the expression as $\frac{(1/.265)}{138}$.

PROBLEMS — **ANSWERS**

PROBLEMS	ANSWERS
1. $\frac{1}{7}$	.143
2. $\frac{1}{35.2}$	.0284
3. $\frac{1}{.1795}$	5.57
4. $\frac{1}{6430}$	.0001555
5. $\frac{1}{\pi}$	.318
6. $\frac{1}{.00417}$	240

THE CF/π AND DF/π SCALES

It should be understood that the use of the CF and DF scales does not increase the power of the instrument to solve problems. In the hands of an experienced computer, however, these scales are used to reduce the number of certain settings. In this way the speed can be increased and errors minimized. When π on the C scale is opposite the right index of the D scale, about half the slide projects beyond the rule. If this part were cut off and used to fill in the opening at the left end, the result would be a "folded" C scale, or CF scale. Such a scale is printed at the top of the slide. It begins at π and the index is near the middle of the rule. The DF scale is similarly placed. Any setting of C on D is automatically set on CF and DF. Thus if 3 on C is opposite 2 on D, then 3 on CF is also opposite 2 on DF. The CF and DF scales can be used for multiplication and division in exactly the same way as the C and D scales.

The most important use of the CF and DF scales is to avoid resetting the slide. If a setting of the indicator cannot be made on the C or D scale, it can be made on the CF or DF scale.

EXAMPLES:

(a) Find 19.2×6.38. Set left index of C on 19.2 of D. Note that 6.38 on C falls outside the D scale. Hence, move the indicator to 6.38 on the CF scale, and read the result 122.5 on the DF scale. Or set the index of CF on 19.2 of DF. Move indicator to 6.38 on CF and read 122.5 on DF.

(b) Find $\frac{8.39 \times 9.65}{5.72}$ Set 5.72 on C opposite 8.39 on D. The indicator cannot be moved to 9.65 of C, but it can be moved to this setting on CF and the result, 14.15, read on DF. Or the entire calculation may be done on the CF and DF scales.

These scales are also helpful in calculations involving π and $1/\pi$. When the indicator is set on any number N on D, the reading on DF is $N\pi$. This can be symbolized as $(DF) = \pi(D)$. Then $(D) = \frac{(DF)}{\pi}$. This leads to the following simple rule.

***Rule:* If the diameter of a circle is set on D, the circumference may be read immediately on DF, and conversely.**

EXAMPLES:

(a) Find 5.6π. Set indicator over 5.6 on D. Read 17.6 under hairline on DF.

(b) Find $8/\pi$. Set indicator over 8 on DF. Read 2.55 under hairline on D.

(c) Find the circumference of a circle whose diameter is 7.2. Set indicator on 7.2 of D. Read 22.6 on DF.

(d) Find the diameter of a circle whose circumference is 121. Set indicator on 121 of DF. Read 38.5 on D.

Finally, these scales are useful in changing radians to degrees and conversely. Since π radians = 180 degrees, the relationship may be written as a proportion $\frac{r}{d} = \frac{\pi}{180}$, or $\frac{r}{\pi} = \frac{d}{180}$.

Rule: **Set 180 of C opposite π on D. To convert radians to degrees, move indicator to r (the number of radians) on DF, read d (the number of degrees) on CF; to convert degrees to radians, move indicator to d on CF, read r on DF.**

There are also other convenient settings as suggested by the proportion. Thus one can set the ratio $\pi/180$ on the CF and DF scales and find the result from the C and D scales.

EXAMPLES:

(a) The numbers 1, 2, and 7.64 are the measures of three angles in radians. Convert to degrees. Set 180 of C on π of D. Move indicator to 1 on DF, read 57.3° on CF. Move indicator over 2 of DF, read 114.6°. Move indicator to 7.64 of DF. Read 437° on CF.

(b) Convert 36° and 83.2° to radians. Use the same setting as in (a) above. Locate 36 on CF. Read 0.628 radians on DF. Locate 83.2 on CF. Read 1.45 radians on DF.

PROBLEMS	ANSWERS
1. 1.414×7.79	11.02
2. 2.14×57.6	123.3
3. $\frac{84.5 \times 7.59}{36.8}$	17.43
4. $2.65 \times \pi$	8.33
5. $\frac{.1955 \times 23.7}{50.7 \times \pi}$	.0291

THE CIF SCALE

Like the other special scales the CIF scale does not increase the power of the instrument to solve problems. It is used to reduce the number of settings or to avoid the awkwardness of certain settings. In this way the speed can be increased and errors minimized.

The CIF scale is a folded CI scale. Its relationship to the CF and DF scales is the same as the relation of the CI scale to the C and D scales.

EXAMPLES:

(a) Find 68.2×1.43. Set the indicator on 68.2 of the D scale. Observe that if the left index is moved to the hairline the slide will project far to the right. Hence merely move 14.3 on CI under the hairline and read the result 97.5 on D at the C index.

(b) Find $2.07 \times 8.4 \times 16.1$. Set indicator on 2.07 on C. Move slide until 8.4 on CI is under hairline. Move hairline to 16.1 on C. Read 280 on D under hairline. Or, set the index of CF on 8.4 of DF. Move indicator to 16.1 on CF, then move slide until 2.07 on CIF is under hairline. Read 280 on DF above the index of CF. Or set 16.1 on CI opposite 8.4 on D. Move indicator to 2.07 on C, and read 280 on D. Although several other methods are possible, the first method given is preferable.

THE CF/M AND DF/M SCALES

The scales CF/M and DF/M are a special feature of PICKETT N4 DUAL BASE LOG LOG SLIDE RULES. With these scales on the rule, logarithms to base 10 may be read from the ordinary C or D scales, and "Natural" logarithms (to base e) may be read directly (with the same setting of the indicator) from these special CF/M and DF/M scales.

The CF/M and DF/M scales are folded at $1/M=2.30$, where the modulus $M=\log_{10}e$. The symbol /M is used to distinguish these scales from the ordinary CF and DF scales which are folded π. The index 1 of CF/M and DF/M is about two-thirds of the length of the rule from the left end.

Although these scales have a special purpose, they may be used in ordinary multiplication, division, and other calculations in exactly the same way that the CF/π and DF/π scales are used. This means, in effect, that PICKETT DUAL BASE LOG LOG SLIDE RULES not only have ordinary C and D scales on both sides of the rule, but also have a set of folded scales on each side.

EXAMPLES: (a) 19.2×6.38. Set left index of C on 19.2 of D. Note that 6.38 on C falls outside the D scale. Hence, move the indicator to 6.38 on the CF/M scale, and read the result 122.5 on the DF/M scale. Observe that the index of the CF/M scale is automatically set at 19.2 of the DF/M scale.

(b) Find $\frac{8.39\times9.65}{5.72}$ Set 5.72 on C opposite 8.39 on D. The indicator cannot be moved to 9.65 of C, but it can be moved to this setting on CF/M and the result, 14.15, read on the DF/M scale. Or the entire calculation may be done on the CF/M and DF/M scales.

THE A AND B SCALES: Square Roots and Squares

When a number is multiplied by itself the result is called the ***square*** of the number. Thus 25 or 5×5 is the square of 5. The factor 5 is called the ***square root*** of 25. Similarly, since $12.25 = 3.5 \times 3.5$, the number 12.25 is called the square of 3.5; also 3.5 is called the square root of 12.25. Squares and square roots are easily found on a slide rule.

Square Roots: To find square roots the A and D scales or the B and C scales are used.

Rule: **The square root of any number located on the A scale is found below it on the D scale.**

Also, the square root of any number located on the B scale (on the slide) is found on the C scale (on the reverse side of the slide).

EXAMPLES: Find the $\sqrt{4}$. Place the hairline of the indicator over 4 on the left end of the A scale. The square root, 2, is read below on the D scale. Similarly the square root of 9 (or $\sqrt{9}$) is 3, found on the D scale below the 9 on the left end of the A scale.

Reading the Scales: The A scale is a contraction of the D scale itself. The D scale has been shrunk to half its former length and printed twice on the same line. To find the square root of a number between 0 and 10 the left half of the A scale is used (as in the examples above). To find the square root of a number between 10 and 100 the right half of the A scale is used. For example, if the hairline is set over 16 on the right half of the A scale (near the middle of the rule), the square root of 16, or 4, is found below it on the D scale.

In general, to find the square root of any number with an odd number of digits or zeros (1, 3, 5, 7, . . .), the left half of the A scale is used. If the number has an even number of digits or zeros (2, 4, 6, 8, . . .), the right half of the A scale is used. In these statements it is assumed that the number is not written in standard form.

The table below shows the number of digits or zeros in the number N and its square root, and also whether right or left half of the A scale should be used.

	ZEROS				or	DIGITS				
	L R	L R	L R	L	R	L R	L R	L R	L R	
N	7 or 6	5 or 4	3 or 2	1	0	1 or 2	3 or 4	5 or 6	7 or 8	etc.
$\sqrt{N}$	3	2	1	0	0	1	2	3	4	etc.

This shows that numbers from 1 up to 100 have one digit in the square root; numbers from 100 up to 10,000 have two digits in the square root, etc. Numbers which are less than 1 and have, for example, either two or three zeros, have only one zero in the square root. Thus $\sqrt{0.004} = 0.0632$, and $\sqrt{0.0004} = 0.02$.

EXAMPLES:

(a) Find $\sqrt{248}$. This number has 3 (an *odd* number) digits. Set the hairline on 248 of the left A scale. Therefore the result on D has 2 digits, and is 15.75 approximately.

(b) Find $\sqrt{563000}$. The number has 6 (an *even* number) digits. Set the hairline on 563 of the right A scale. Read the figures of the square root on the D scale as 75. The square root has 3 digits and is 750 approximately.

(c) Find $\sqrt{.00001362}$. The number of *zeros* is 4 (an *even* number.) Set the hairline on 1362 of the right half of the A scale. Read the figures 369 on the D Scale. The result has 2 zeros, and is .00369.

If the number is written in standard form, the following rule may be used. If the exponent of 10 is an even number, use the left half of the A scale and multiply the reading on the D scale by 10 to an exponent which is ½ the original. If the exponent of 10 is an odd number, move the decimal point one place to the right and decrease the exponent of 10 by one, then use the right half of the A scale and multiply the reading on the D scale by 10 to an exponent which is ½ the reduced exponent. This rule applies to either positive or negative exponents of 10.

EXAMPLES:

(1) Find the square root of 3.56×10^4. Place hairline of indicator on 3.56 on the left half of the A scale and read 1.887 on the D scale. Then the square root is $1.887 \times 10^2 = 188.7$.

(2) Find the square root of 7.43×10^{-5}. Express the number as 74.3×10^{-6} Now place the hairline of the indicator over 74.3 on the right half of the A scale and read 8.62 on the D scale. Then the desired square root is 8.62×10^{-3}.

All the above rules and discussion can be applied to the B and C scales if it is more convenient to have the square root on the slide rather than on the body of the rule.

***Squares: To find the square of a number, reverse the process* for finding the square root. Set the indicator over the number on the D scale and read the square of that number on the A scale; or set the indicator over the number on the C scale and read the square on the B scale.**

EXAMPLES:

(a) Find $(1.73)^2$ or 1.73×1.73. Locate 1.73 on the D scale. On the A scale find the approximate square 3.

(b) Find $(62800)^2$. Locate 628 on the D scale. Find 394 above it on the A scale. The number has 5 digits. Hence the square has either 9 or 10 digits. Since, however, 394 was located on the right half of the A scale, the square has the *even* number of digits, or 10. The result is 3,940,000,000.

(c) Find $(.000254)^2$. On the A scale read 645 above the 254 of the D scale. The number has 3 zeros. Since 645 was located on the side of the A scale for "odd zero" numbers, the result has 7 zeros, and is 0.0000000645.

PROBLEMS	ANSWERS
1. $\sqrt{7.3}$	2.7
2. $\sqrt{73}$	8.54
3. $\sqrt{841}$	29
4. $\sqrt{0.062}$	0.249
5. $\sqrt{0.00000094}$	0.00097
6. $(3.95)^2$	15.6
7. $(48.2)^2$	2320
8. $(0.087)^2$	0.00757
9. $(0.00284)^2$	0.00000807
10. $(635000)^2$	4.03×10^{11}

THE K SCALE: Cube Roots and Cubes

Just below the D scale on the back of the rule is a scale marked with the letter K; this scale may be used in finding the cube or cube root of any number.

Rule: **The cube root of any number located on the K scale is found directly above on the D scale.**

EXAMPLE: Find the $\sqrt[3]{8}$. Place the hairline of the indicator over the 8 at the left end of the K scale. The cube root, 2, is read directly above on the D scale.

Reading the scales: The cube root scale is directly below the D scale and is a contraction of the D scale itself. The D scale has been shrunk to one third its former length and printed three times on the same line. To find the cube root of any number between 0 and 10 the left third of the K scale is used. To find the cube root of a number between 10 and 100 the middle third is used. To find the cube root of a number between 100 and 1000 the right third of the K scale is used to locate the number.

In general to decide which part of the K scale to use in locating a number, mark off the digits in groups of three starting from the decimal point. If the left group contains one digit, the left third of the K scale is used; if there are two digits in the left group, the middle third of the K scale is used; if there are three digits, the right third of the K scale is used. In other words, numbers containing 1, 4, 7, $\cdots$ digits are located on the left third; numbers containing 2, 5, 8, $\cdots$ digits are located on the middle third; and numbers containing 3, 6, 9, $\cdots$ digits are located on the right third of the K scale. The corresponding number of digits or zeros in the cube roots are shown in the table below and whether the left, center or right section of the K scale should be used.

	ZEROS				or	DIGITS			
	L C R	L C R	L C R	L C	R	L C R	L C R	L C R	L C R
N	11, 10, 9	8, 7, 6	5, 4, 3	2, 1	0	1, 2, 3	4, 5, 6	7, 8, 9	10, 11, 12
$\sqrt[3]{N}$	3	2	1	0	0	1	2	3	4

EXAMPLES:

(a) Find $\sqrt[3]{6.4}$. Set hairline over 64 on the left most third of the K scale. Read 1.857 on the D scale.

(b) Find $\sqrt[3]{64}$. Set hairline over 64 on the middle third of the K scale. Read 4 on the D scale.

(c) Find $\sqrt[3]{640}$. Set hairline over 64 on the right most third of the K scale. Read 8.62 on the D scale.

(d) Find $\sqrt[3]{6,400}$. Set hairline over 64 on the left third of the K scale. Read 18.57 on the D scale.

(e) Find $\sqrt[3]{64,000}$. Set hairline over 64 on the middle third of the K scale. Read 40 on the D scale.

(f) Find $\sqrt[3]{0.0064}$. Use the left third of the K scale, since the first group of three, or 0.006, has only one non-zero digit. The D scale reading is then 0.1857.

(g) Find $\sqrt[3]{0.064}$. Use the middle third of the K scale, reading 0.4 on D.

If the number is expressed in standard form it can either be written in ordinary form or the cube root can be found by the following rule.

Rule: Make the exponent of 10 a multiple of three, and locate the number on the proper third of the K scale. Read the result on the D scale and multiply this result by 10 to an exponent which is ⅓ the former exponent of 10.

Examples: Find the cube root of 6.9×10^3. Place the hairline over 6.9 on the left third of the K scale and read 1.904 on the D scale. Thus the desired cube root is 1.904×10^1. Find the cube root of 4.85×10^7. Express the number as 48.5×10^6 and place the hairline of the indicator over 48.5 on the middle third of the K scale. Read 3.65 on the D scale. Thus the desired cube root is 3.65×10^2. or 365. Find the cube root of 1.33×10^{-4}. Express the number as 133×10^{-6} and place the hairline over 133 on the right third of the K scale. Read 5.10 on the D scale. The required cube root is 5.10×10^{-2}.

Cubes: **To find the cube of a number, reverse the process for finding cube root. Locate the number on the D scale and read the cube of that number on the K scale.**

EXAMPLES:

(a) Find $(1.37)^3$. Set the indicator on 1.37 of the D scale. Read 2.57 on the K scale.

(b) Find $(13.7)^3$. The setting is the same as in example (a), but the K scale reading is 2570, or 1000 times the former reading.

(c) Find $(2.9)^3$ and $(29)^3$. When the indicator is on 2.9 of D, the K scale reading is 24.4. The result for 29^3 is therefore 24,400.

(d) Find $(6.3)^3$. When the indicator is on 6.3 of D, the K scale reading is 250.

PROBLEMS:	ANSWERS:
1. 2.45^3	14.7
2. 56.1^3	176,600
3. $.738^3$	.402
4. 164.5^3	4,451,000
5. $.0933^3$	.000812
6. $\sqrt[3]{5.3}$	1.744
7. $\sqrt[3]{71}$	4.14
8. $\sqrt[3]{815}$	9.34
9. $\sqrt[3]{.0315}$	.316
10. $\sqrt[3]{525,000}$	80.7
11. $\sqrt[3]{.156}$	.538

COMBINED OPERATIONS

Many problems involve expressions like $\sqrt{ab}$, or $(a \sin \theta)^2$, etc. With a little care, many such problems involving combined operations may be easily computed. The list of possibilities is extensive, and it is no real substitute for the thinking needed. to solve them. Consequently, only a few examples will be given.

The A and B scales may be used for multiplication or division in exactly the same way as the C and D scales. Since the scales are shorter, there is some loss in accuracy. Nevertheless, most computers employ the A and B scales (in conjunction with the C and D scales) to avoid extra steps which would also lead to loss of accuracy.

EXAMPLE:

(a) Find $\sqrt{3.25 \times 4.18}$. First find the product 3.25×4.18 using the left A and B scales. Set left index of B on 3.25 of A. Move indicator to 4.18 of B. Read the square root of this product under the hairline on D. The result is 3.69 approximately.

(b) Find 1.63×5.41^2. Set the left index on B under 1.63 of A. Move the indicator to 5.41 on C. Read the result 47.7 under the hairline on A.

(c) Find $5^{\frac{3}{2}}$ or $5^{1.5}$. This is the same as $(\sqrt{5})^3$. Hence set the indicator on 5 of the left A scale. Read 11.2 on the middle K scale under the hairline.

(d) Find $24^{\frac{2}{3}}$ or $(\sqrt[3]{24})^2$. Set the indicator on 24 of the middle K scale. Read the result after squaring as 8.3 on the A scale under the hairline.

THE $\sqrt{}$ SCALES: Square Roots and Squares

When a number is multiplied by itself the result is called the *square* of the number. Thus 25 or 5×5 is the square of 5. The factor 5 is called the *square root* of 25. Similarly, since $12.25 = 3.5\times3.5$, the number 12.25 is called the square of 3.5; also 3.5 is called the square root of 12.25. Squares and square roots are easily found on a slide rule.

Square Root. Just below the D scale is another scale marked with the square root symbol, $\sqrt{}$.

***Rule.* The square root of any number located on the D scale is found directly below it on the $\sqrt{}$ scale.**

EXAMPLES: Find $\sqrt{4}$. Place the hairline of the indicator over 4 on the D scale. The square root, 2, is read directly below. Similarly, the square root of 9 (or $\sqrt{9}$) is 3, found on the $\sqrt{}$ scale directly below the 9 on the D scale.

Reading the Scales. The square root scale directly below the D scale is an enlargement of the D scale itself. The D scale has been "stretched" to double its former length. Because of this the square root scale seems to be cut off or to end with the square root of 10, which is about 3.16. To find the square root of numbers greater than 10 the bottom $\sqrt{}$ scale is used. This is really the rest of the stretched D scale. The small figure 2 near the left end is placed beside the mark for 3.2, and the number 4 is found nearly two inches farther to the right. In fact, if 16 is located on the D scale, the square root of 16, or 4, is directly below it on the *bottom scale* of the rule.

In general, the square root of a number between 1 and 10 is found on the upper square root scale. The square root of a number between 10 and 100 is found on the lower square root scale. If the number has an odd number of digits or zeros (1, 3, 5, 7, ...), the upper $\sqrt{}$ scale is used. If the number has an even number of digits or zeros (2, 4, 6, 8, ...), the lower $\sqrt{}$ scale is used. The first three (or in some cases even four) figures of a number may be set on the D scale, and the first three (or four) figures of the square root are read directly from the proper square root scale.

The table below shows the number of digits or zeros in the number N and its square root.

	ZEROS				or	DIGITS				
	U L	U L	U L	U	L	U L	U L	U L	U L	
N	7 or 6	5 or 4	3 or 2	1	0	1 or 2	3 or 4	5 or 6	7 or 8	etc.
$\sqrt{N}$	3	2	1	0	0	1	2	3	4	etc.

The above table is reproduced on some models of Pickett Slide Rules.

This shows that numbers from 1 up to 100 have one digit in the square root; numbers from 100 up to 10,000 have two digits in the square root, etc. Numbers which are less than 1 and have, for example, either two or three zeros, have only one zero in the square root. Thus $\sqrt{0.004} = 0.0632$, and $\sqrt{0.0004} = 0.02$.

EXAMPLES:

(a) Find $\sqrt{248}$. Set the hairline on 248 of the D scale. This number has 3 (an *odd* number) digits. Therefore the figures in the square root are read from the upper $\sqrt{}$ scale as 1575. The resulth as 2 digits, and is 15.75 approximately.

(b) Find $\sqrt{563000}$. Set the hairline on 563 of the D scale. The number has 6 (an *even* number) digits. Read the figures of the square root on the bottom scale as 75. The square root has 3 digits and is 750 approximately.

(c) Find $\sqrt{.00001362}$. Set the hairline on 1362 of the D scale. The number of *zeros* is 4 (an *even* number). Read the figures 369 on the bottom scale. The result has 2 zeros, and is .00369.

***Squaring* is the opposite of finding the square root. Locate the number on the proper $\sqrt{}$ scale and with the aid of the hairline read the square on the D scale.**

EXAMPLES:

(a) Find $(1.73)^2$ or 1.73×1.73. Locate 1.73 on the √ scale. On the D scale find the approximate square 3.

(b) Find $(62800)^2$. Locate 628 on the √ scale. Find 394 above it on the D scale. The number has 5 digits. Hence the square has either 9 or 10 digits. Since, however, 628 was located on the lower of the √ scales, the square has the *even* number of digits, or 10. The result is 3,940,000,000.

(c) Find $(.000254)^2$. On the D scale read 645 above the 254 of the √ scale. The number has 3 zeros. Since 254 was located on the upper of the √ scales, the square has the odd number of digits, or 7. The result is 0.0000000645.

PROBLEMS:	ANSWERS:	PROBLEMS:	ANSWERS:
1. $\sqrt{7.3}$	2.7	6. $(3.95)^2$	15.6
2. $\sqrt{73}$	8.54	7. $(48.2)^2$	2320
3. $\sqrt{841}$	29	8. $(0.087)^2$	0.00757
4. $\sqrt{0.062}$	0.249	9. $(0.00284)^2$	0.00000807
5. $\sqrt{0.00000094}$	0.00097	10. $(635000)^2$	4.03×10^{11}

THE $\sqrt[3]{\ }$ SCALES: Cube Roots and Cubes

At the top of the rule there is a cube root scale marked $\sqrt[3]{\ }$. It is a D scale which has been stretched to three times its former length, and then cut into three parts which are printed one below the other.

Rule. **The cube root of any number on the D scale is found directly above it on the $\sqrt[3]{\ }$ scales.**

Example: Find the $\sqrt[3]{8}$. Place the hairline of the indicator over the 8 on the D scale. The cube root, 2, is read above on the upper $\sqrt[3]{\ }$ scale.

Reading the scale: To find the cube root of any number between 0 and 10 the upper $\sqrt[3]{\ }$ scale is used. To find the cube root of a number between 10 and 100 the middle $\sqrt[3]{\ }$ scale is used. To find the cube root of a number between 100 to 1000 the lower $\sqrt[3]{\ }$ scale is used.

In general to decide which part of the $\sqrt[3]{\ }$ scale to use in locating the root, mark off the digits in groups of three starting from the decimal point. If the left group contains one digit, the upper $\sqrt[3]{\ }$ scale is used; if there are two digits in the left group, the middle $\sqrt[3]{\ }$ scale is used; if there are three digits, the lower $\sqrt[3]{\ }$ scale is used. Thus, the roots of numbers containing 1, 4, 7,... digits are located on the upper $\sqrt[3]{\ }$ scale; numbers containing 2, 5, 8,... digits are located on the middle $\sqrt[3]{\ }$ scale; and numbers containing 3, 6, 9,... digits are located on the lower $\sqrt[3]{\ }$ scale. The corresponding number of digits or zeros in the cube roots are shown in the table below and whether the upper, middle or lower section of the $\sqrt[3]{\ }$ scale should be used.

	ZEROS				or	DIGITS			
	U M L	U M L	U M L	U M	L	U M L	U M L	U M L	U M L
N	11, 10, 9	8, 7, 6	5, 4, 3	2, 1	0	1, 2, 3	4, 5, 6	7, 8, 9	10, 11, 12
$\sqrt[3]{N}$	3	2	1	0	0	1	2	3	4

The above table is reproduced on some models of Pickett Slide Rules.

EXAMPLES:

(a) Find $\sqrt[3]{6.4}$. Set hairline over 64 on the D scale. Read 1.857 on the upper $\sqrt[3]{\ }$ scale.

(b) Find $\sqrt[3]{64}$. Set hairline over 64 on the D scale. Read 4 on the middle $\sqrt[3]{\ }$ scale.

(c) Find $\sqrt[3]{640}$. Set hairline over 64 on the D scale. Read 8.62 on the lower $\sqrt[3]{\ }$ scale.

(d) Find $\sqrt[3]{6{,}400}$. Set hairline over 64 on the D scale. Read 18.57 on the upper $\sqrt[3]{\ }$ scale.

(e) Find $\sqrt[3]{64{,}000}$. Set hairline over 64 on the D scale. Read 40 on the middle $\sqrt[3]{\ }$ scale.

(f) Find $\sqrt[3]{0.0064}$. Set hairline over 64 on the D scale. Read 0.1857 on the upper $\sqrt[3]{\ }$ scale.

(g) Find $\sqrt[3]{0.064}$. Set hairline over 64 on the D scale. Read 0.4 on the middle $\sqrt[3]{\ }$ scale.

If the number is expressed in standard form it can either be written in ordinary form or the cube root can be found by the following rule.

Rule: **Make the exponent of 10 a multiple of three, and locate the number on the D scale. Read the result on the $\sqrt[3]{\ }$ scale and multiply this result by 10 to an exponent which is 1/3 the former exponent of 10.**

EXAMPLES: Find the cube root of 6.9×10^3. Place the hairline over 6.9 on the D scale and read 1.904 on the upper $\sqrt[3]{\ }$ scale. Thus the desired cube root is 1.904×10^1. Find the cube root of 4.85×10^7. Express the number as 48.5×10^6 and place the hairline of the indicator over 48.5 on the D scale. Read 3.65 on the middle $\sqrt[3]{\ }$ scale. Thus the desired cube root is 3.65×10^2 or 365. Find the cube root of 1.33×10^{-4}. Express the number as 133×10^{-6} and place the hairline over 133 on the D scale. Read 5.10 on the lower $\sqrt[3]{\ }$ scale. The required cube root is 5.10×10^{-2}.

Cubes: To find the cube of a number, reverse the process for finding the cube root. Locate the number on the $\sqrt[3]{\ }$ scale and read the cube of that number on the D scale.

EXAMPLES:

(a) Find $(1.37)^3$. Set the indicator on 1.37 of the $\sqrt[3]{\ }$ scale. Read 2.57 on the D scale.

(b) Find $(13.7)^3$. The setting is the same as in example (a), but the D scale reading is 2570, or 1000 times the former reading.

(c) Find $(2.9)^3$ and $(29)^3$. When the indicator is on 2.9 of the $\sqrt[3]{\ }$ scale, the D scale reading is 24.4. The result for 29^3 is therefore 24,400.

(d) Find $(6.3)^3$. When the indicator is on 6.3 of the $\sqrt[3]{\ }$ scale, the D scale reading is 250.

PROBLEMS:	ANSWERS:
1. 2.45^3	14.7
2. 56.1^3	176,600
3. $.738^3$	.402
4. 164.5^3	4,451,000
5. $.0933^3$	.000812

PROBLEMS:	ANSWERS:
6. $\sqrt[3]{5.3}$	1.744
7. $\sqrt[3]{71}$	4.14
8. $\sqrt[3]{815}$	9.34
9. $\sqrt[3]{.0315}$	.316
10. $\sqrt[3]{525{,}000}$	80.7
11. $\sqrt[3]{.156}$	.538

The Ln Scale: Logarithms to base *e*, Powers of *e*.

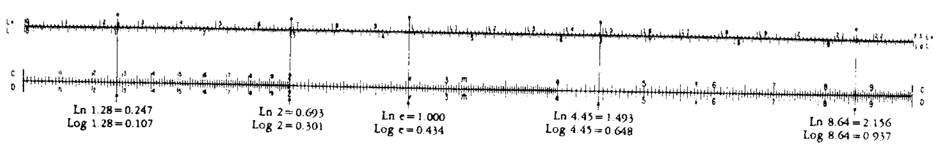
The Ln scale is similar to the regular L scale. It is used for problems with base *e*. For many problems it is more convenient to use Ln than the Log Log scales on advanced models. In particular, it enables you to solve problems with *powers of e* in combined operations. Its range (from 0 to 2.3) is greater than the range of the L scale (from 0 to 1). The inclusion of the Ln scale completes the DUAL-BASE features of the Pickett Slide Rules.

By computing a "characteristic" you can use the Ln scale to find any power of *e*; thus the effective range for powers of *e* is from 0 to infinity. Since powers of *e* are read on the C (or D) scale, accuracy to 3 or 4 significant figures is obtained no matter how large or how small the numbers are. The Ln scale saves steps in many computational problems.

In this section you will learn how to use the Ln scale to find powers of *e* and logarithms to base *e*. The Ln scale is a uniformly divided scale similar to the L scale. It is used with problems in base *e* just as the L scale is used with problems in base 10.

When you study the charts in this manual always set the slide and cursor on your rule like the ones pictured in the manual. Study the *slide rule* itself along with the charts. In this manual we will use the symbol Ln x for the logarithm of x to base *e*.

(a) Finding logarithms.



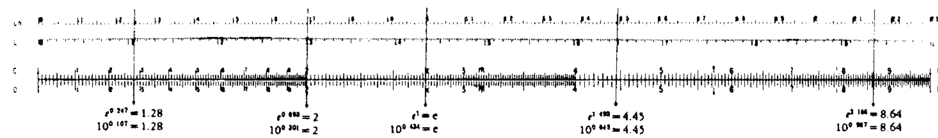
***Rule:* To find mantissas of logarithms: Set the number on the C (or D) scale, and read the mantissa for base 10 from L, or for base *e* from Ln. If L scales are on slide, set on C. If L scales are on body, set on D. Characteristic for base 10 is found by usual method. "Characteristic" for base *e* is explained in Section (e). Also, see below.**

**For $1 \leqq x \leqq 10$, we have $0 \leqq \log x \leqq 1$.
For $1 \leqq x \leqq 10$, we have $0 \leqq \text{Ln}\, x \leqq 2.30258$
For the same domain ($1 \leqq x \leqq 10$), the range of Ln is greater than the range of L.**

- EXAMPLES:

1. Ln 1.473 = 0.387	2. Log 1.473 = 0.168
3. Ln 2.34 = 0.850	4. Log 2.34 = 0.369
5. Ln π = 1.145	6. Ln 3.49 = 1.250
7. Ln 4 = 1.386	8. Ln 4.62 = 1.530
9. Ln 5.17 = 1.643	10. Ln 7.35 = 1.995
11. Ln 8.05 = 2.086	12. Ln 9 = 2.197
13. Ln 3.62 = 1.286	14. Log 3.62 = 0.559
15. Ln 1.91 = 0.647	16. Log 1.91 = 0.281
17. Ln 2.66 = 0.978	18. Log 2.66 = 0.425

***(b) Powers of* e *and of* 10.** In the figure below, notice that the cursor hairlines shown are in the same positions as in Section (a).



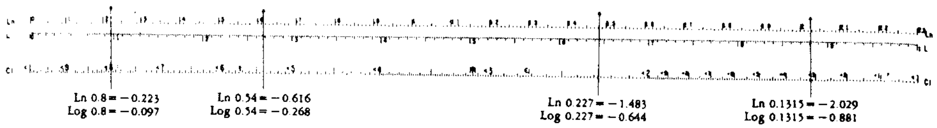
***Rule:* To find powers of e and of 10. Set the exponent of *e* on Ln, or of 10 on L, and read the power on C (or D). If L scales are on slide, use C; if they are on body, use D. The decimal point of the answer is found by special rules. See Section (f). Also see below.**

**For $0 \leqq y \leqq 1$, we have $1 \leqq 10^y \leqq 10$.
For $0 \leqq y \leqq 2.30258$, we have $1 \leqq e^y \leqq 10$.**

- EXAMPLES:

1. $\sqrt{e} = e^{0.5} = 1.649$	2. $e^{0.81} = 2.25$
3. $\sqrt{10} = 10^{0.5} = 3.16$	4. $e^{1.15} = 3.16$
5. $e^{1.36} = 3.90$	6. $e^{1.63} = 5.104$
7. $e^{1.875} = 6.52$	8. $e^2 = 7.39$
9. $e^{2.138} = 8.48$	10. $e^{2.252} = 9.51$
11. $10^{0.732} = 5.40$	12. $10^{0.9} = 7.95$
13. $\sqrt{e^3} = e^{1.5} = 4.48$	14. $10^{0.405} = 2.54$
15. $e^{0.1} = 1.105$	16. $e^{0.205} = 1.228$
17. $e^{0.532} = 1.702$	18. $e^{0.01} = 1.0101$
19. $e^{0.99} = 2.691$	20. $e^{1.01} = 2.746$

(c) *Finding logarithms of proper fractions.* The logarithm of each proper fraction is a negative number. It is written in two ways; for example, log 0.5= −0.301 or 9.699-10; also, Ln 0.5= −0.693 or 9.307-10. For slide rule work the form 9.— — — -10 is not needed.



***Rule:* To find mantissas of logarithms of numbers between 0.1 and 1, set number on CI (or DI), read mantissa for base 10 on L, for base e on Ln. If L scales are on the slide, use CI. If L scales are on the body, use DI. For smaller numbers, see Section (f). Also, see below.**

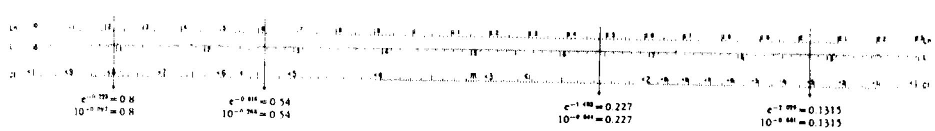
For $-1 \leq y \leq 0$, we have $0.1 \leq 10^y \leq 1.0$.
For $-2.30258 \leq y \leq 0$, we have $0.1 \leq e^y \leq 1.0$.

For the same domain ($0.1 \leq x \leq 1$), the range of Ln is greater than the range of L. For x in this domain, the logarithm is read *directly* from the scale and written with the negative sign. For x not in this domain, the characteristic must be found by a special rule.

- EXAMPLES:

1. Ln 0.15 = −1.897	2. Ln 0.1625 = −1.818
3. Ln 0.19 = −1.661	4. Log 0.19 = −0.721
5. Ln 0.202 = −1.599	6. Log 0.202 = −0.695
7. Ln 0.259 = −1.351	8. Log 0.259 = −0.587
9. Ln 0.3 = −1.204	10. Ln π/10 = −1.158
11. Ln 0.85 = −0.163	12. Ln 0.92 = −0.083
13. Log 0.742 = −0.130	14. Ln 0.742 = −0.298
15. Log 0.363 = −0.440	16. Ln 0.363 = −1.014
17. Log 0.178 = −0.750	18. Ln 0.178 = −1.726
19. Ln 0.120 = −2.120	20. Log 0.12 = −0.921
21. Ln 0.103 = −2.27	22. Log 0.103 = −2.274

(d) *Powers for negative exponents.* For negative exponents, powers are all less than 1. Hence they are proper fractions. In the figure below, notice that the cursor hairlines are shown in the same positions as in Section (c), above.



***Rule:* To find powers of e and of 10 for negative exponents, set the exponent of e on Ln, or the exponent of 10 on L, and read the power on CI (or DI). If L scales are on slide, use CI; if they are on the body, use DI. The decimal point in the answer is found by special rules. See Section (f). Also, see below.**

For $-1 \leq y \leq 0$, we have $0.1 \leq 10^y \leq 1.0$.
For $-2.30258 \leq y \leq 0$, we have $0.1 \leq e^y \leq 1.0$.

Although the domain of y is greater for base e, the range of 10^y and e^y is the same. The exponents for this range may be set directly on the Ln or the L scale.

- EXAMPLES:

1. $e^{-2} = 0.135$	2. $10^{-0.8} = 0.1585$
3. $e^{-1} = 0.368$	4. $10^{-0.1} = 0.794$
5. $e^{-0.2} = 0.819$	6. $e^{-0.63} = 0.533$
7. $e^{-0.27} = 0.763$	8. $e^{-1.54} = 0.214$
9. $e^{-1.27} = 0.281$	10. $e^{-2.08} = 0.125$
11. $e^{-2.29} = 0.1013$	12. $e^{-1.65} = 0.192$
13. $10^{-0.67} = 0.2138$	14. $e^{-0.72} = 0.487$
15. $10^{-0.44} = 0.363$	16. $e^{-0.56} = 0.571$
17. $10^{-0.25} = 0.562$	18. $e^{-0.99} = 0.372$
19. $e^{-1.5} = 0.223$	20. $10^{-0.15} = 0.708$

(e) *Finding the Characteristic* **Base 10**

The characteristic is the exponent of 10 when the number is expressed in *standard form.*

***Rule:* To express a number in standard form: (i) place a decimal point at the right of the first non-zero digit, (ii) start at the right of the first non-zero digit in the original number and count the digits and zeros passed over in reaching the decimal point. The result of the count is the numerical value of the characteristic, or exponent of 10. If the original decimal point is toward the right, the characteristic is positive (+). If the original decimal point is toward the left, the characteristic is negative (−). Indicate that the result of (i) is multiplied by 10 with this exponent.**

- EXAMPLES:

Number	Number in standard form	Characteristic
1. 5,790,000	5.79×10^6	6
2. 0.000283	2.83×10^{-4}	−4
3. 44	4.4×10^1	1
4. 0.623	6.23×10^{-1}	−1
5. 8.15	8.15×10^0	0
6. 461,328	4.61328×10^5	5
7. 0.0000005371	5.371×10^{-7}	−7
8. 0.0306	3.06×10^{-2}	−2

Base e

The term "characteristic" as used here will mean the number to which a reading from the Ln scale must be added to account for logarithms not in its range.

Rule: **First express the number in standard form. Read the logarithm of the first factor directly from Ln, as in Section (a). Multiply 2.30258 by the exponent of 10 in the second factor. If the exponent is positive, add this result to the direct reading. If the exponent is negative, subtract the result from the direct reading.**

• EXAMPLES:

Verify that the logarithm to base e for the examples at the left is as follows:

1. $\text{Ln } 5.79 \times 10^6 = 1.756 + 6(2.303) = 15.574$
2. $\text{Ln } 2.83 \times 10^{-4} = 1.040 - 4(2.303) = -8.170$
3. $\text{Ln } 4.4 \times 10^1 = 1.482 + 2.303 = 3.785$
4. $\text{Ln } 6.23 \times 10^{-1} = 1.829 - 2.303 = -0.474$
5. $\text{Ln } 8.15 \times 10^0 = 2.098 - 0 = 2.098$
6. $\text{Ln } 4.61 \times 10^5 = 1.528 + 5(2.303) = 13.041$
7. $\text{Ln } 5.371 \times 10^{-7} = 1.681 - 7(2.303) = -14.437$
8. $\text{Ln } 3.06 \times 10^{-2} = 1.118 - 2(2.303) = -3.487$

For $0 < x < \infty$, we have $-\infty < \log x < +\infty$.
For $0 < x < \infty$, we have $-\infty < \text{Ln } x < +\infty$.

(f) *Extending the range for 10^y and e^y.*

Base 10

For y not in the interval between 0 and 1, the standard method of finding 10^y first expresses it as the product of two factors. Thus $10^{2.5} = 10^2 \times 10^{0.5}$. One factor has an integral exponent. The other factor has a fractional exponent in the interval 0 to 1. The second factor is computed by the methods of Section (b), and Section (d). The first factor then determines the position of the decimal point in the final answer.

• EXAMPLES:

1. $10^{2.5} = 10^2 \times 10^{0.5} = 10^2 \times 3.16 = 316.$
2. $10^{4.26} = 10^4 \times 10^{0.26} = 10^4 \times 1.82 = 18{,}200.$
3. $10^{-5.38} = 10^{-5} \times 10^{-0.38} = 10^{-5} \times 0.417 = 0.00000417.$
4. $10^{-2.71} = 10^{-2} \times 10^{-0.71} = 10^{-2} \times 0.195 = 0.00195.$

For $-\infty < y < +\infty$, we have $0 < 10^y < +\infty$.
The effective range for powers of 10 is infinite.
Three or four significant figures of e^y can be found.

• EXAMPLES:

1. $10^{3.916} = 8{,}240.$ 2. $10^{-3.916} = 0.0001214$
3. $10^{5.023} = 105{,}400.$ 4. $10^{-5.023} = 0.000{,}009{,}48$
5. $10^{14.622} = 4.19 \times 10 = 419{,}000{,}000{,}000{,}000.$
6. $10^{-23.877} = 0.000{,}000{,}000{,}000{,}000{,}000{,}000{,}001{,}327$

Base *e*

For y not in the interval $0 \leqq y \leqq 2.30258$, express e as the product of two factors. For example, $e^{3.5} = e^{2.3} \times e^{1.2} = 10e^{1.2}$. One method of finding these factors is to divide the

exponent y by 2.30258 (or a rounded value of this divisor, such as 2.303), to determine an integral quotient, q, and a remainder, r. Then $y = 2.303q + r$, and $e^y = e^{2.303q + r} = e^{2.303q} \times e^r = (e^{2.303})^q \times e^r = 10^q \times e^r$.

The value of the second factor is computed by the methods of Section (b), and Section (d). The first factor is used to determine the decimal point.

• EXAMPLES:

1. To find $e^{6.54}$, first divide 6.54 by 2.303, obtaining quotient 2 and remainder 1.934. Then $e^{6.54} = 10^2 \times e^{1.934}$. Set cursor to 1.934 on Ln. Read 6.92 on C (or D). Then answer is $100 \times 6.92 = 692$.
2. Find $e^{-6.54}$. As in Example 1, $e^{-6.54} = 10^{-2} \times e^{-1.934}$. Set cursor hairline to 1.934 on Ln. Read 0.114 on CI (or DI). Then $e^{-6.54} = 0.00144$.
3. Find $e^{17.4}$. Divide 17.4 by 2.303, obtaining quotient ("characteristic") 7 and remainder 1.28. Set hairline to 1.28 of Ln. Read 360 on C (or D). Multiply by 10^7.

For $-\infty < y < +\infty$, we have $0 < e^y < +\infty$. The effective range for powers of e is infinite. Three or four significant figures of e^y can be found.

(g) *A short cut in using Ln 10=2.30258*

To extend the range of Ln the number 2.30258 is needed. Suppose that, to save work, the number 2.3 is used. Some error will of course occur. For example, the remainder in division will be too large. How can we easily correct for this error? The following simple rule will serve:

Rule: **Take 1 percent of the quotient and divide it by 4. Subtract the result from the remainder to obtain the correct remainder to set on Ln.**

• EXAMPLES:

Find $e^{17.4}$ (Compare with Example 3, Section f, under Base e). *Divide:*

$$\begin{array}{r} 7. \\ 2.3\overline{)17.4} \\ \underline{16.1} \\ 1.3 \end{array} \quad \text{or} \quad \begin{array}{r} 7. \\ 2.30258\overline{)17.40000} \\ \underline{16.11806} \\ 1.28194 \end{array}$$

Take 1% of 7; $0.01 \times 7 = 0.07$; Divide by 4. $0.07 \div 4 = 0.02$, approx. Subtract 0.02 from 1.3, to obtain 1.28, the corrected remainder. Then $e^{17.4} = e^{1.28} \times 10^7$. The basis of this rule is explained below.

Consider $x = e^n$. Divide n by 2.30258, and denote the integral part of the quotient by q and the remainder by r. Then, $n = 2.30258q + r$, $r < 2.30258$:

We now propose to use 2.3 as divisor in place of 2.30258. We require the quotient to again be q, but get a new remainder which we denote by R, where $R > r$. Then, $n = 2.3q + R$, where $R < 2.3$.

Subtracting this from the former equation, we have $0 = 0.00258q + r - R$, or $r = R - 0.00258q$.

Thus the error, $R - r$, in the remainder is $0.00258q$. If this is rounded off to $0.0025q$, it expresses one-fourth of 1 percent of the quotient.

When the slide rule is used to divide by 2.30, proceed as follows: Set 2.30 of C over 17.4 of D. Under 1 of C read 7.56 on D. The integral part, or "characteristic," is 7. Multiply the decimal fraction 0.56 by 2.3, using the C and D scales. Obtain 1.29 as the reduced exponent of e.

With Model 4 rules the quotient may be obtained by merely setting the exponent on DF/M and reading the quotient on D. The relation between readings on the D and the DF/M scales may be indicated symbolically as follows:

$$(D)\times 2.30=(DF/M) \text{ and } (DF/M)\div 2.30=(D).$$

For some purposes and for some exponents, this slide rule method is not sufficiently accurate.

- EXAMPLES:

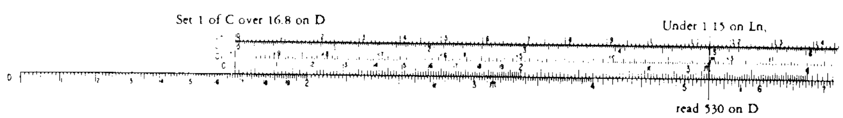
1. Find $e^{7.61}$. Divide 7.61 by 2.3; quotient 3, remainder 0.71. Correction is 0.03/4=.01. Hence $e^{7.61}=e^{0.70}\times 10^3$, or 2,018.
2. Find $e^{-6.95}$. Divide 6.95 by 2.3 to get characteristic 3 and remainder 0.05. Correction is 0.0 3/4=0.0075. Corrected remainder is 0.0425. Hence $e^{-6.95}=e^{-0.0425}\times 10^{-3}$. Set 0.0425 on Ln, read 0.958 on CI. Point off 3 places to the left, to get 0.000958.
3. Find e^9. Divide 9 by 2.3. Characteristic is 3, remainder 2.1. Correction is 0.03/4=0.0075, or .01. $e^9=e^{2.09}\times 10^3$. Set 2.09 on Ln, read 8.1 on C. Then e^9=8100.
4. Find e^{-9}, or $e^{-2.09}\times 10^{-3}$. Set 2.09 on Ln. Read on 0.125 on CI, point off 3 places to left to find e^{-9}=0.000123.

The L Scale: Logarithms to Base 10

In this section you will learn how the L and Ln scales are used in combination with other scales. The methods used when the L scales are on the body differ from those used when they are on the slide. Follow only the instructions for the type of slide rule you have.

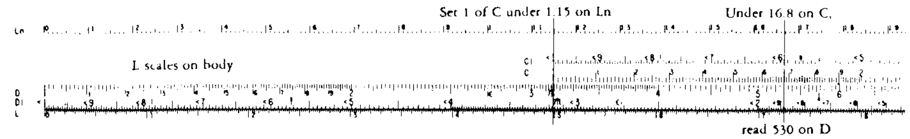
In solving problems, first express the numbers in standard form as explained in Section (f) and Section (g). The calculations are carried through within the ranges of L and Ln provided, and the decimal points are determined by special rules.

(h) Multiplication with powers. The scales below are set to find $16.8\times e^{1.15}$ with L scales on the slide.



Notice that when 1 of the C scale is set over 16.8 of the D scale, the product of 16.8 and any number set on C is read on D. But by setting the cursor hairline over 1.15 of Ln the value of $e^{1.15}$ is *automatically* set on C. This number (actually 3.16) does not have to be read. The product is on D. With the log log scales, this value (3.16) must be read and transferred to C before the multiplication can be started.

***Rule* for $a.e^y$. If L scales are on slide, set 1 of C over a on D. Move cursor hairline to y of Ln. Read figures of answer on D. Determine the decimal point by standard form method. If L scales are on body, *begin* with e^y. Set cursor hairline to y on L. Set 1 of C under cursor hairline. Move cursor to a of C. Read answer on D. With powers of 10, use L in the same way.**



(i) Division with powers. Remember that division is the opposite of multiplication. The scales pictured in Section (h), on page 11, are set to divide 530 by $e^{1.15}$; that is, to find $530/e^{1.15}$ using D and C, or 530 $e^{-1.15}$, using D and CI.

L Scales on Slide

***Rule:* To divide a/e^y, set y on Ln over a on D. Under 1 of C read a/e^y on D.**

(j) Examples for practice.

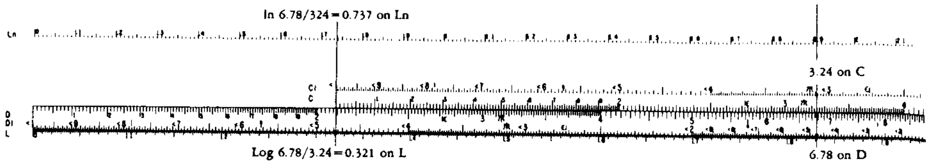
1. Find 2.79 $e^{1.945}$/3.82. Set hairline over 2.79 on D. Move slide so 3.82 on C is under hairline. Move hairline to 1.945 on Ln. Read 5.12 on D.
2. Find 17.35 $e^{1.226}$ sin 43°. Set 1 of C over 17.35 on D. Move hairline over 1.226 on Ln. Move right index of C under hairline. Move hairline to 43 on S. Read 40.3 on D.
3. Find 0.0000452$e^{7.61}$ (see Ex. 1, p. 10). Write the work in standard form: $4.52\times 10^{-5}\times e^{0.70}\times 10^3=4.52\times 10^{-2}$. Set index of C over 452 on D. Move hairline over 0.70 on Ln. Read 912 on D. Answer is 0.0912.
4. Find $5.27^2e^{12.7}$. First rewrite $e^{12.7}$ as $e^{1.19}\times 10^5$. Set hairline over 5.27 on√. Turn rule over, and set index of C under hairline. Move hairline to 1.19 on Ln. Read 910 on D. Note 5.27^2 is about 30 or, roughly, 3×10. Also $e^{1.19}$ is about 3. Answer, then, is about $3\times 3\times 10\times 10^5$ or 9×10^6. Correct to three significant figures, answer is 9.10×10^6.

L Scales on Body

***Rule:* To divide a/e^y, set 1 of C under y of Ln. Move hairline over a on D. Read a/e^y on C under the hairline.**

1'. Find 2.79 $e^{1.945}$/3.82. Set hairline over 1.945 of Ln. Set slide so 3.82 of C is under hairline. Move hairline over 2.79 on C. Read 5.12 on D.
2'. Find 17.35$e^{1.226}$ sin 43°. Set hairline over 1.226 on Ln. Move slide so 17.35 on CI is under hairline. Move hairline to 43 on S. Read 40.3 on D.
3'. Find 0.0000452e^{761} (see Ex. 1, p. 10). Write the work in standard form: $4.52\times 10^{-5}\times 3^{0.70}\times 10^3=4.52\times e^{0.70}\times 10^{-2}$.
Set hairline over 0.70 on Ln. Move slide so index of C is under hairline. Move hairline over 452 on C. Read 912 on D. Answer is 0.0912.
4'. Find $5.27^2e^{12.7}$. First rewrite $e^{12.7}$ as $e^{1.19}\times 10^5$. Set hairline over 1.19 on Ln. It is not convenient to use the A scale for 5.27^2. Move slide so 527 on CI is under hairline. Move hairline to 527 on C. Read 910 on D. Note 5.27^2 is about 30, or, roughly, 3×10. Also $e^{1.19}$ is about 3. Answer, then, is about $3\times 3\times 10\times 10^5$ or 9×10^6. Correct to three significant figures, answer is 9.10×10^6.

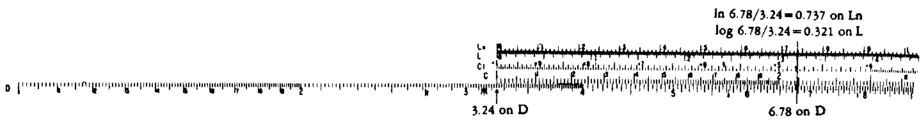
(k) Logarithms of combined opera ns. The scales below are set to find ln 6.78/3.24 and log 6.78/3.24 with L scales on the body.



First, divide 6.78 by 3.24 in the usual way using C and D scales. Move cursor over 1 of C. Read ln 6.78/3.24 on Ln and read log 6.78/3.24 on L. If L scales are on the slide, close rule (move slide so C and D indexes coincide) before reading from Ln or L. Or, if you prefer, first set slide so 1 of C is over 3.24 on D. Move hairline over 6.78 on D. Read logarithm from Ln or L.

***Rule:* To find ln *a*/*b* or log *a*/*b*, set *b* on C over *a* on D. At index of C read ln on Ln and log on L. Characteristics must be found by a special rule.**

If L scales are on slide, set 1 of C over *b* on D. Move indicator over *a* on D. Read logarithm from Ln on L. This method requires only one setting of the slide.



(l) Powers of other bases. Sometimes powers of bases other than 10 or *e* are needed. If a slide rule has Log Log scales, they may be used to find these powers within the range provided. If the slide rule does not have Log Log scales, or if the power is outside the scale range provided one of the following methods may be used.

Using Base *e*.

Examples

1. Find $1.5^{2.4}$. Write $e^x=1.5$. Set 1.5 on C (or D), find x=0.405 on Ln. Then $(1.5)^{2.4}=(e^{0.405})^{2.4}=e^{0.972}$ by multiplying the exponents. Using Ln again, set 0.972 on Ln, read the answer 2.65 on C (or D). This solution can be expressed in logarithmic form as follows: ln $1.5^{2.4}=2.4$ ln 1.5.
2. Find $18.5^{-6.37}$. Set $y=18.5^{-6.37}$. Then ln y=—6.37×ln 18.5. Now ln 18.5= ln 1.85×10=0.615+2.303 or 2.918. Now —6.37×2.918=18.58 and 18.58÷2.3 =8 with remainder of 0.18. But 1% of 8=.08, and .08÷4=.02, so with the correction 0.18—0.02=0.16, we have to find $e^{-0.16}\times10^{-8}$. Set hairline over 0.16 on Ln, read 0.852 on CI (or DI). The quotient 8 tells us the answer is 0.852×10^{-8} $=8.52\times10^{-9}$.
3. Find $0.88^{0.25}$. Set $y=0.88^{0.25}$. Then ln y=0.25×ln 0.88. Write ln 0.88=ln 8.8× 10^{-1}. Set hairline over 8.8 of C (or D), read 2.175 on Ln. Then ln $8.8\times10^{-1}=2.175$ —2.303=—0.128, and 0.25×(—0.128)=—0.032. Set hairline over 0.032 on Ln, read answer 0.968 on CI (or DI).

Using Base 10.

Examples

1. Find $1.5^{2.4}$. Write $10^x=1.5$. Set 1.5 on C (or D), find x=0.176 on L. Then $(1.5)^{2.4}=(10^{0.176})^{2.4}=10^{0.422}$, by multiplying exponents. Set 0.422 on L, read the answer 2.65 on C (or D). This solution can be expressed in logarithmic form as follows: log $1.5^{2.4}=2.4$ log 1.5.

2. Find $18.5^{-6.37}$. Set $y=18.5^{-6.37}$. Then log $y=-6.37\times\log 18.5$. Now log 18.5= 0.267+1=1.267. Now —6.37×1.267=—8.07. We must now set the hairline over 0.07 on L, *reading from right to left*, or subtract 8.07 from 10.00—10 to write the logarithm with a positive mantissa, namely 1.93−10. Set the hairline over 0.93 on L and read 8.50 on C (or D). Then the result is 8.50×10^{-9}.
3. Find $0.88^{0.25}$. Set $y=0.88^{0.25}$. Then log $y=0.25\times\log 0.88$. Write log 0.88=log 8.8×10^{-1}. Set hairline over 8.8 on C (or D), read 0.944 on L. Then log $88\times10^{-1}=$ 0.944−1, or 3.944−4.0.25 (3.944−4)=0.986−1. Set hairline over 0.986 of L, read 9.68 on C (or D). Then answer is 0.968.

(m) Hyperbolic functions. The Ln scale is very helpful in finding values of the hyperbolic functions. This is especially true for models which do not provide Log Log scales or hyperbolic function scales. However, even with Model 4 on which these extra scales are available, the Ln scale simplifies the work in problems that fall outside the range of the scales provided.

By definition, sinh $x=(e^x-e^{-x})/2$, or sinh $x=(e^x/2)-(e^{-x}/2)$.

By definition, cosh $x=(e^x+e^{-x})/2$, or cosh $x=(e^x/2)+(e^{-x}/2)$.

By definition, tanh $x=(e^x-e^{-x})/(e^x+e^{-x})=(\sinh x)/(\cosh x)$.

***Rule:* To find sinh *x* or cosh x, set hairline over *x* on Ln, read e^x on C (or D) and e^{-x} on CI (or DI). For sinh *x*, subtract e^{-x} from e^x and divide the result by 2. For cosh *x*, add e^x and e^{-x}, and divide by 2. To find tanh *x*, use Ln to find e^x and e^{-x}; divide their difference by their sum.**

***Rule:* For x > 3, sinh *x*=cosh *x*=e /2 can be found by setting the index of the C scale over 5 on the D scale, moving the hairline to x on Ln, and reading the result on D.**

Examples.

1. Find sinh 5.4 or $e^{5.4}/2$. Divide 5.4 by 2.303, obtaining quotient 2 and remainder 0.794. $(e^{5.4}/2)=(e^{0.794}/2)\ 10^2$. Set right hand index of C over 5 of D. Move hairline to 0.794 on Ln. Read 1.107 on D. This must be multiplied by 10^2, so sinh 5.4 =110.7.
2. Find sinh $24=e^{24}/2$. First write $e^{24}=e^{0.975}\times10^{10}$. Using Ln, find $e^{0.975}=2.65$. Then 2.65/2=1.32, so sinh $24=1.32\times10^{10}$.

Extending the ranges. Remember that if $x>2.3$, you must first divide x by 2.3 and correct the remainder. The corrected remainder is set on Ln instead of *x*, and the integral quotient q is the exponent of 10 such that the factor 10^q determines the position of the decimal point.

For $x>3$, we have $(e^{-x}/2)<0.025$. Hence, for $x>3$, sinh x=cosh $x=e^x/2$, approximately, and tanh $x=1$. For $x<0.10$, we have sinh $x=x$, cosh $x=1$, and tanh $x=x$, approximately.

On Model 4 the values of e^x and $e^{-x}$for x < 23 can be found directly on the Log Log scales by using the DF/M scale. For $x>10$ the accuracy is poor. For $x>23$, the Log Log scales are useless.

Examples.

3. Find cosh $4.8=e^{4.8}/2$. Write $e^{4.8}=e^{0.195}\times10^2$. Set left index of C over 5 on D. Move hairline over 0.195 on Ln. Read $e^{0.195}/2=0.608$ on D. Then cosh 4.8=60.8.
4. Find tanh 1.3. Using Ln, read $e^{1.3}=3.67$ on C (or D) and $e^{-1.3}=0.273$ on CI (or DI). Then 3.67−0.273=3.397 and 3.67+0.273=3.94; hence tanh 1.3=0.862.

(n) *Applied problems.*

1. As an extraordinary example consider the following quotation:
"The total N for the entire line is N=0.1118×2000=223.6 nepers, and the ratio of input to output current

$I_s/I_r = e^{223.6} \doteq 10^{97}$"*

Calculate $e^{223.6}$.

By the method of Section (g), above, we divide 223.6 by 2.3, and correct the remainder.

$$\begin{array}{r} 97 \\ 23\overline{)223.60} \\ \underline{207.} \\ 16.6 \\ \underline{16.1} \\ .50 \end{array}$$

$0.01 \times 97 = 0.97$

$1/4 \times .97 = 0.24$

$0.50 - .24 = 0.26$

Set hairline over 0.26 of Ln.
Read 1.297 on C.
The result is 1.297×10^{97}. The result found by logarithms is 1.286×10^{97}. The error is 0.85%, or under 1%, and occurs because the correction formula uses 0.0025 instead of 0.00258. This shows that the method using Ln is sufficiently accurate for all exponents up to 100; such large exponents are exceedingly rare.

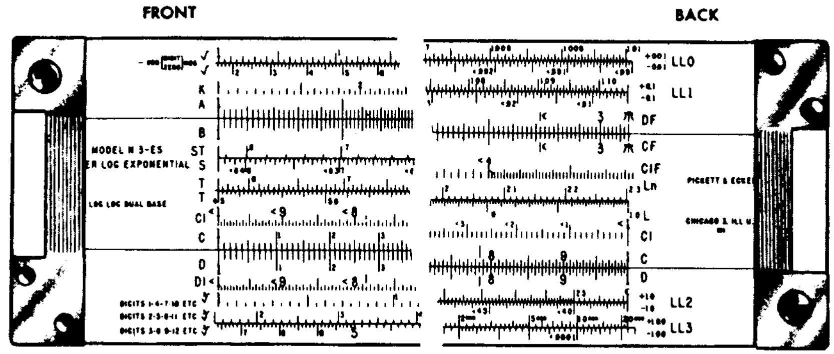
2. A table of standard sizes for rectangular wire may be made by inserting 38 geometric means between the diameter (0.46 in.) of Gauge 0000 and the diameter (0.005 in.) of Gauge 36 of the American Wire Gauge.

Calculate the common ratio $r = \sqrt[39]{\frac{0.4600}{0.005}}$, and compute the 36th term.

First note that $r=(460/5)^{1/39}$, or $(92)^{1/39}$. Write 10 =92. Set 92 on C (or D), find mantissa of x, or 0.964 on L. Then $x=1.964$. Then $r=(10)^{1.964/39}=10^{0.0503}$; set 0.0503 on L, read r=1.123 on C (or D). The 36th term is 0.005×1.123^{35}, or $0.005 \times 10^{35} \times (0.0503) = 0.005 \times 10^{1.761}$. Set hairline over 0.761 on L, read 5.77 on C (or D). Finally, compute $0.005 \times 10 \times 5.77 = 0.289$ in., or 289 mils, approximately.

3. The formula for the current in a certain circuit is $i=1.25\ (1-e^{-80t})$, $0 \leqq t \leqq 0.01$. Find i for t=0.006; that is, $i=1.25\ (1-e^{-80 \times 0.006}) = 1.25\ (1-e^{-0.48})$.
Set hairline over 0.48 on Ln, read $e^{-0.48}=0.619$ on CI (or DI). Then $i=1.25\ (1-0.619) = 1.25 \times 0.381 = 0.476$.

4. In a problem similar to 3, above, the formula is $i=4\ (1-e^{-40t}), 0 \leqq t \leqq 0.02$. Find i for t=0.015; that is, $i=4\ (1-e^{-0.60})$. Answer: 1.804.



THE EXTRA SECTIONS OF THE LOG LOG SCALE

Model 3 provides an extra section of Log Log scales. This section is used in exactly the same way as the rest of the LL scales (LL1, LL2, LL3) and requires no additional explanation. Note that it is possible to set numbers near 1 to great accuracy; thus 1.00333, which is a six figure number, is easily set. An example which illustrates the added convenience of having this scale is given below.

Example 1. Find $1.0261^{0.342}$. Set the left index of the C scale opposite 1.0261 of LL1+. Move the hairline over 342 of C, and read the result on the LL0+ scale as 1.00886. 0.1

Note that if the LL0+ scale were not on the slide rule, the *answer could not be read directly*. In that case (that is, with slide rules in which the lower bound of the range of the LL scales is 1.01 instead of 1.001), the result may be computed by using logarithms, but with less accuracy.

Thus, set the hairline over 1.0261 of the LL1+ scale, and the left index of the C scale under the hairline. The reading on the D scale gives $\log_e 1.0261$, which is 0.0258. Move the hairline over 0.342 of C (which multiplies 0.0258 by 0.342), and read 0.0088 on the D scale. This is $\log_e 1.261^{0.342}$. Now $\log_e (1 + X) = X$, approximately, if X is sufficiently small. In this example, let X = 0.0088 (which is small), and then the number, or 1 + X, is 1.0088.

With Model 3 it is unnecessary to resort to this longer procedure.

The readings on the LL0− scale are reciprocals of the readings on the LL0+ scale. This scale extends the LL1− scale from 0.99 to 0.999 which is, of course, much closer to 1. This scale is used in the same way that the LL1−, LL2−, and LL3 scales are used. No additional explanation beyond that given in the manual for these scales should be needed.

The S, T, and ST Scales: TRIGONOMETRY

The branch of mathematics called *trigonometry* arose historically in connection with the measurement of triangles. However, it now has many other uses in various scientific fields.

Some important formulas from trigonometry are listed here for ready reference.

The trigonometric ratios may be defined in terms of a right triangle as follows:

Sine of angle A $= \dfrac{\text{side opposite}}{\text{hypotenuse}}$ (written Sin A $= \dfrac{a}{h}$);

Cosine of angle A $= \dfrac{\text{side adjacent}}{\text{hypotenuse}}$ (written Cos A $= \dfrac{b}{h}$);

Tangent of angle A $= \dfrac{\text{side opposite}}{\text{side adjacent}}$ (written Tan A $= \dfrac{a}{b}$);

Cotangent of angle A $= \dfrac{\text{side adjacent}}{\text{side opposite}}$ (written Cot A $= \dfrac{b}{a}$);

Secant of angle A $= \dfrac{\text{hypotenuse}}{\text{side adjacent}}$ (written Sec A $= \dfrac{h}{b}$);

Cosecant of angle A $= \dfrac{\text{hypotenuse}}{\text{side opposite}}$ (written Cosec A $= \dfrac{h}{a}$).

These ratios are functions of the angle. The definitions may be extended to cover cases in which the angle A is not an interior angle of a right triangle, and hence may be greater than 90 degrees. Note that the sine and cosecant are reciprocals, as are the cosine and secant, and the tangent and cotangent. Therefore,

$$\text{Sin A} = \frac{1}{\text{Cosec A}}, \text{ and Cosec A} = \frac{1}{\text{Sin A}};$$

$$\text{Tan A} = \frac{1}{\text{Cot A}}, \text{ and Cot A} = \frac{1}{\text{Tan A}};$$

$$\text{Cos A} = \frac{1}{\text{Sec A}}, \text{ and Sec A} = \frac{1}{\text{Cos A}}.$$

When the sum of two angles equals 90°, the angles are *complementary.*

$$\begin{aligned} \text{Sin A} &= \cos(90° - \text{A}) \\ \text{Cos A} &= \sin(90° - \text{A}) \\ \text{Tan A} &= \cot(90° - \text{A}) \\ \text{Cot A} &= \tan(90° - \text{A}) \end{aligned}$$

When the sum of two angles equals 180°, the angles are *supplementary.*

$$\begin{aligned} \sin(180° - \text{A}) &= \text{Sin A} \\ \cos(180° - \text{A}) &= -\text{Cos A} \\ \tan(180° - \text{A}) &= -\text{Tan A} \end{aligned}$$

The following laws are applicable to any triangle.

$$\text{A} + \text{B} + \text{C} = 180°$$

$$\text{Law of sines: } \frac{\text{Sin A}}{a} = \frac{\text{Sin B}}{b} = \frac{\text{Sin C}}{c}$$

$$\text{Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos \text{A}$$

THE S SCALE: Sines and Cosines

The scale marked S is used in finding the approximate sine or cosine of any angle between 5.7 degrees and 90 degrees, Since $\sin x = \cos(90 - x)$, the same graduations serve for both sines and cosines. Thus sin 6° = cos (90° − 6°) = cos 84°. The numbers printed at the right of the longer graduations are read when sines are to be found. Those printed at the left are used when cosines are to be found. On the slide rule, angles are divided decimally instead of into minutes and seconds. Thus sin 12.7° is represented by the 7th small graduation to the right of the graduation marked 78|12.

Sines (or cosines) of all angles on the S scale have no digits or zeros—the decimal point is at the left of figures read from the C (or D) scale.

***Rule:* To find the sine of an angle on the S scale, set the hairline on the graduation which represents the angle. (Remember to read sines from left to right and the numbers to the right of the graduation are for sines). Read the sine on the C scale under the hairline. If the slide is placed so the C and D scales are exactly together, the sine can also be read on the D scale, and the mantissa of the logarithm of the sine (log sin) may be read on the L scale.**

EXAMPLE:

(1) Find sin 15° 30′ and log sin 15°30′. Set left index of C scale over left index of D scale. Set hairline on 15.5° (i.e., 15°30′) on S scale. Read sin 15.5° = .267 on C or D scale. Read mantissa of log sin 15.5° = .427 on L scale. According to the rule for characteristics of logarithms, this would be 9.427 − 10.

***Rule:* To find the cosine of an angle on the S scale, set the hairline on the graduation which represents the angle. (Remember to read cosines from right to left and the numbers to the left of the graduation are for cosines). Read the cosine on the C scale under the hairline. If the slide is placed so the C and D scales are exactly together, the cosine can also be read on the D scale, and the mantissa of the cosine (log cos) may be read on the L scale.**

EXAMPLE:

(1) Find cos 42° 15′ and log cos 42° 15′. Set left index of C scale over left index of D scale. Set hairline on 42.25° (i.e., 42° 15′) on S scale. Read cos 42.25° = .740 on C or D scale. Read mantissa of log cos 42.25° = .869 on L scale. According to rule for characteristics of logarithms, this would be 9.869 − 10.

Finding the Angle

If the value of trigonometric ratio is known, and the size of the angle less than 90° is to be found, the above rules are reversed. The value of the ratio is set on the C scale, and the angle itself read on the S scale.

EXAMPLES:

(a) Given $\sin x = .465$, find x. Set indicator on 465 of C scale, read $x = 27.7°$ on the S scale.

(b) Given $\cos x = .289$, find x. Set indicator on 289 on C scale. Read $x = 73.2°$ on the S scale.

PROBLEMS:	ANSWERS:
1. Sin 9.6°	.167
2. Sin 37.2°	.605
3. Sin 79.0°	.982
4. Cos 12.2°	.977
5. Cos 28.6°	.878
6. Cos 37.2°	.794
7. Cosec 15.8°	3.68
Note: $\text{Cosec}\ \theta = \frac{1}{\sin\theta}$	
8. Sec. 19.3°	1.060
Note: $\text{Sec}\ \theta = \frac{1}{\cos\theta}$	
9. $\text{Sin}\ \theta = .1737$	$\theta = 10°$
10. $\text{Sin}\ \theta = .98$	$\theta = 78°$
11. $\text{Sin}\ \theta = .472$	$\theta = 28.2°$
12. $\text{Cos}\ \theta = .982$	$\theta = 10.8°$
13. $\text{Cos}\ \theta = .317$	$\theta = 71.5°$
14. $\text{Cos}\ \theta = .242$	$\theta = 76°$
15. $\text{Sec}\ \theta = 1.054$ $(\sec\theta = \frac{1}{\cos\theta})$	$\theta = 18.7°$
16. $\text{Cosec}\ \theta = 1.765$ $(\text{cosec}\ \theta = \frac{1}{\sin\theta})$	$\theta = 34.5°$
17. log sin 10.4°	9.256−10
18. log sin 24.2°	9.613−10
19. log cos 14.3°	9.986−10
20. log cos 39.7°	9.886−10
21. $\log\sin\theta = 9.773-10$	$\theta = 36.4°$
22. $\log\sin\theta = 9.985-10$	$\theta = 75.°$
23. $\log\cos\theta = 9.321-10$	$\theta = 77.9°$
24. $\log\cos\theta = 9.643-10$	$\theta = 63.9°$

THE T SCALE: Tangents and Cotangents

The T scale, together with the C or CI scales, is used to find the value of the tangent or cotangent of angles between 5.7° and 84.3°. Since $\tan x = \cot(90-x)$, the same graduations serve for both tangents and cotangents. For example, if the indicator is set on the graduation marked 30, the corresponding reading on the C scale is .577, the value of tan 30°. This is also the value of cot 60°, since $\tan 30° = \cot(90° - 30°) = \cot 60°$. Moreover, $\tan x = 1/\cot x$; in other words, the tangent and cotangent of the same angle are reciprocals. Thus for the same setting, the reciprocal of cot 60°, or 1/.577, may be read on the CI scale as 1.732. This is the value of tan 60.

A single T scale reading from 5.7° to 45° (left to right) and from 45° to 84.3° (right to left) will enable you to make all calculations. In order to provide ease in reading and to simplify the solution of certain problems the T scale on some models of slide rules is doubled. That is, the scale for tangents of angles from 5.7° to 45° is above the line and the scale for angles from 45° to 84.3° is below the line. Check your slide rule and determine the section of the manual that is applicable.

For single T scale

Rule. **Set the angle value on the T scale and read**
(i) tangents of angles from 5.7° to 45° on C,
(ii) tangents of angles from 45° to 84.3° on CI,
(iii) cotangents of angles from 45° to 84.3° on C,
(iv) cotangents of angles from 5.7° to 45° on CI.

If the slide is set so that the C and D scales coincide, these values may also be read on the D scale. Care must be taken to note that the T scale readings for angles between 45° and 84.3° *increase from right to left.*

In case (i) above, the tangent ratios are all between 0.1 and 1.0; that is, the decimal point is at the left of the number as read from the C scale.

In case (ii), the tangents are greater than 1.0, and the decimal point is placed to the right of the first digit as read from the CI scale. For the cotangent ratios in cases (iii) and (iv) the situation is *reversed.* Cotangents for angles between 45° and 84.3° have the decimal point at the left of the number read from the C scale. For angles between 5.7° and 45° the cotangent is greater than 1 and the decimal point is to the right of the first digit read on the CI scale. These facts may be summarized as follows.

Rule: **If the tangent or cotangent ratio is read from the C scale, the decimal point is at the *left* of the first digit read. If the value is read from the CI scale, it is at the *right* of the first digit read.**

EXAMPLES:

(a) Find $\tan x$ and $\cot x$ when $x = 9°50'$. First note that $50 = \frac{50}{60}$ of 1 degree = .83°, approximately. Hence $9°\,50' = 9.83°$. Locate $x = 9.83°$ on the T scale. Read $\tan x = .173$ on the C scale, and read $\cot x = 5.77$ on the CI scale.

(b) Find $\tan x$ and $\cot x$ when $x = 68.6°$. Locate $x = 68.6°$ on the T scale reading from right to left. Read 255 on the CI scale. Since all angles greater than 45° have tangents greater than 1 (that is, have one digit as defined above), $\tan x = 2.55$. Read $\cot 68.6° = .392$ on the C scale.

Finding the Angle

If the value of the trigonometric ratio is known, and the size of the angle less than 90° is to be found, the above rules are reversed. The value of the ratio is set on the C or CI scale, and the angle itself read on the T scale.

EXAMPLES:

(c) Given $\tan x = .324$, find x. Set 324 on the C scale, read 17.9° on the T scale.

(d) Given $\tan x = 2.66$, find x. Set 266 on the CI scale, read $x = 69.4°$ on the T scale.

(e) Given $\cot x = .630$, find x. Set 630 on the C scale, read $x = 57.8°$ on the T scale.

(f) Given $\cot x = 1.865$, find x. Set 1865 on the CI scale, read 28.2° on the T scale.

For double T scale

***Rule.* Set the angle x on the T scale: (i) above the line if $5.7 \le x \le 45°$, and (ii) below the line if $45° \le x \le 84.3°$, and read the value of the tangent on the C scale, and cotangent on the CI scale.**

In case (i), the decimal point of the tangent is at the left of the first digit read on C. In case (ii), the decimal point of the tangent is at right of the first digit read on C. In case (i), the decimal point of the cotangent is at the right of the first digit read on CI. In case (ii), the decimal point is at the left of the first digit read on CI.

EXAMPLES:

(a) Find tan 14.7° and cot 14.7°. Set indicator over 14.7 on upper T scale. Read tan 14.7° = 0.262 on C, and cot 14.7° = 3.81 on CI.

(b) Find tan 72.3° and cot 72.3°. Set indicator over 72.3 on lower T scale. Read tan 72.3° = 3.13 on C and cot 72.3 = 0.319 on CI.

PROBLEMS:	ANSWERS:
1. tan 18.6°	.337
2. tan 66.4°	2.29
3. cot 31.7°	1.619
4. cot 83.85°	.1078
5. $\tan \theta = 1.173$	$\theta = 49.55°$
6. $\cot \theta = .387$	$\theta = 68.84°$

THE ST SCALE: Small Angles

The sine and the tangent of angles of less than about 5.7° are so nearly equal that a single scale, marked ST, may be used for both. The graduation for 1° is marked with the degree symbol (°). To the left of it the primary graduations represent tenths of a degree. The graduation for 2° is just about in the center of the slide. The graduations for 1.5° and 2.5° are also numbered.

Rule: **For small angles, set the indicator over the graduation for the angle on the ST scale, then read the value of the sine or tangent on the C scale. Sines or tangents of angles on the ST scale have one zero.**

EXAMPLES:

(a) Find sin 2° and tan 2°. Set the indicator on the graduation for 2° on the ST scale. Read sin 2° = .0349 on the C scale. This is also the value of tan 2° correct to three digits.

(b) Find sin 0.94° and tan 0.94°. Set the indicator on 0.94 of ST. Read sin 0.94° = tan 0.94° = .0164 on the C scale.

Since $\cot x = 1/\tan x$, the cotangents of small angles may be read on the CI scale. Moreover, tangents of angles between 84.3° and 89.42° can be found by use of the relation $\tan x = \cot (90 - x)$. Thus $\cot 2° = 1/\tan 2° = 28.6$, and $\tan 88° = \cot 2° = 28.6$. Finally, it may be noted that $\csc x = 1/\sin x$, and $\sec x = 1/\cos x$. Hence the value of these ratios may be readily found if they are needed. Functions of angles greater than 90° may be converted to equivalent (except for sign) functions in the first quadrant.

EXAMPLES:

(a) Find cot 1.41° and tan 88.59°. Set indicator at 1.41° on ST. Read cot 1.41° = tan 88.59° = 40.7 on CI.

(b) Find csc 21.8° and sec 21.8°. Set indicator on 21.8° of the S scale. Read csc 21.8° = 1/sin 21.8° = 2.69 on CI. Set indicator on 68.2° of the S scale (or 21.8 reading from right to left), and read sec 21.8° = 1.077 on the CI scale.

When the angle is less than 0.57° the approximate value of the sine or tangent can be obtained directly from the C scale by the following procedure.

Read the ST scale as though the decimal point were at the left of the numbers printed, and read the C scale (or D, CI, etc.) with the decimal point one place to the left of where it would normally be. Thus sin 0.2° = 0.00349; tan 0.16° = 0.00279, read on the C scale.

Two seldom used special graduations are also placed on the ST scale. One is indicated by a longer graduation found just to the left of the graduation for 2° at about 1.97°. When this graduation is set opposite any number of minutes on the D scale, the sine (or the tangent) of an angle of that many minutes may be read on the D scale under the C index.

Sin 0° = 0, and sin 1′ = .00029, and for small angles the sine increases by .00029 for each increase of 1′ in the angle. Thus sin 2′ = .00058; sin 3.44′ = .00100, and the sines of all angles between 3.44′ and 34.4′ have two zeros. Sines of angles between 34.4′ and 344′ (or 5.73°) have one zero. The tangents of these small angles are very nearly equal to the sines.

EXAMPLE: Find sin 6′. With the hairline set the "minute graduation" opposite 6 located on the D scale. Read 175 on the D scale under the C index. Then sin 6′ = .00175.

The second special graduation is also indicated by a longer graduation located at about 1.18°. It is used in exactly the same way as the graduation for minutes. Sin 1″ = .0000048, approximately, and the sine increases by this amount for each increase of 1″ in the angle, reaching .00029 for sin 60″ or sin 1′ = .00029.

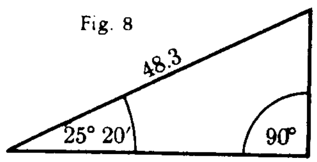
Trigonometric Computations

Many formulas involve both trigonometric ratios and other factors. By using several different scales such computations are easily done.

EXAMPLES:

(a) Find the length of the legs of a right triangle in which the hypotenuse is 48.3 ft. and one acute angle is 25° 20′.

Fig. 8

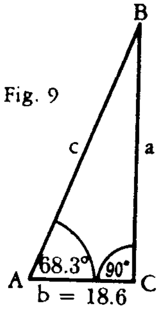


The side opposite the given acute angle is equal to 48.3 sin 25° 20′. Hence we compute 48.3 × sin 25.3°. Set the index (right-hand index in this example) of the C scale on 48.3 of the D scale. Move the hairline over 25.3° on the S scale. Read 20.7 under the hairline on the D scale. Another method is to set the left index of the C scale and D scale opposite each other. Set the hairline over 25.3° on the S scale. Move the slide so that (right) index of the C scale is under the hairline. Read 20.7 on the D scale under 48.3 of the C scale. The length of the other leg is equal to 48.3 cos 25.3° or 48.3 sin 64.7° = 43.7.

(b) One angle of a right triangle is 68.3°, and the adjacent side is 18.6 ft. long. Find the other side and the hypotenuse.

a = 18.6 tan 68.3° or 18.6/cot 68.3
c = 18.6/cos 68.3°.

Fig. 9



For single T scale

To find *a*, set the indicator on 18.6 of the D scale, pull the slide until 68.3 of the T scale (read from right to left) is under the hairline, and read *a* = 46.7 on the D scale under the right index of the C scale. To find c, pull the slide until 68.3° of the S scale (read from right to left) is under the hairline (which remains over 18.6), and read the result 50.3 on the D scale at the right index.

For double T scale

To find *a*, set the left index of C on 18.6 of the D scale. Move the hairline over 68.3 on the lower T scale and read *a* = 46.7 on the D scale under the hairline. To find C, move the hairline over 18.6 of the D scale. Pull the slide until 68.3° of the S scale (read from right to left) is under the hairline, and read the result 50.3 on the D scale at the right index.

This problem may also be solved by the law of sines, namely,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \quad \text{or} \quad \frac{\sin 68.3}{a} = \frac{\sin 21.7}{18.6} = \frac{1}{c}$$

Set 21.7° on S opposite 18.6 on D. Read c = 50.3 on D under 1 of C. Move indicator to 68.3° on S, read 46.7 under the hairline on D.

(c) Find one side and two angles of an obtuse triangle when two sides and an included angle are known.

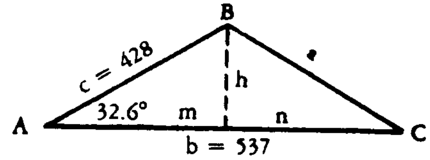


Fig. 10

Given: c = 428; b = 537; A = 32.6°

Find: a, B, and C.

Construct: Line h from the vertex of B perpendicular to AC. This divides △ ABC into 2 right triangles.

Then h = c Sin A, from the formula for sine of an angle.
Set right index of C over 428 on D and read h = 231 on D scale opposite 32.6° on S scale. Also,

$$m = \frac{h}{\text{Tan } A}, \text{ from formula for tangent of an angle.}$$

Opposite h (231) on D scale set 32.6° on T scale, and read *m* = 361 on D scale opposite right index on C scale. Since *m* is now known,
n = b—m = 176.

$$\text{Tan } C = \frac{h}{n}, \text{ from formula for tangent of an angle.}$$

Opposite 231 on D scale set 176 on C scale and read 1.312, or tan C on D opposite left index of C scale.
For single T scale set 1.312 on CI (see rule page 30), and read C = 52.7° on T.

For double T scale set 1.312 on C and read C = 52.7° on lower T.

$$\text{Finally, } a = \frac{h}{\text{Sin } C}, \text{ from formula for sine of an angle.}$$

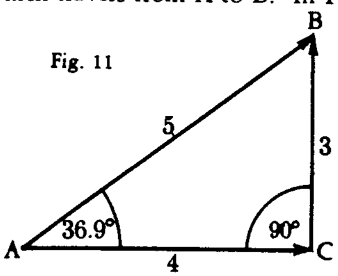
Set hairline over 231 on D. Move 52.7° on S under hairline.
Read a = 290 on D at right index of C.
Since A + B + C = 180°,
then B = 180° − (A+C) = 180° − 32.6° − 52.7° = 94.7°.

PART 3—ELEMENTARY VECTOR METHODS

COMPLEX NUMBERS AND VECTORS

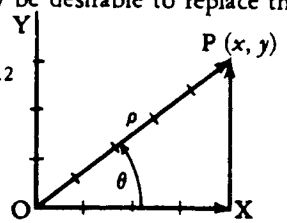
A *vector* quantity is one which has both *magnitude* and *direction*. For example, force and velocity are vector quantities. A quantity which has magnitude only is called a *scalar*. For example, mass is a scalar. Vector quantities are often represented by directed straight line segments. The length of the segment represents the magnitude in terms of a selected scale unit. The segment has an initial point A and a terminal point B, and direction is usually indicated by an arrowhead at B pointing in the same direction as the motion of a point which travels from A to B. In Fig. 11, three vectors are represented; namely AB of magnitude 5, AC of magnitude 4, and CB of magnitude 3. Vectors AB and AC have the same initial point, A, and form an angle, CAB, of 36.9°. The initial point of vector CB is at the terminal point of AC. Vectors CB and AB have the same terminal point.

Fig. 11



Operations with vectors (for example, addition and multiplication) are performed according to special rules. Thus in Fig. 11, AB may be regarded as the *vector sum* of AC and CB. AB is called the *resultant* of AC and CB; the latter are *components* of AB, and in this case are at right angles to each other. It is frequently desirable to express a given vector in terms of two such components at right angles to each other. Conversely, when the components are given, it may be desirable to replace them with the single resultant vector.

Fig. 12



In algebra, the complex number $x + iy$, where $i = \sqrt{-1}$, is represented by a point P (x, y) in the complex plane, using a coordinate system in which an axis of "pure imaginary" numbers, OY, is at right angles to an axis of "real" numbers, OX.

The same point can be expressed in terms of polar coordinates (ρ, θ) in which the radius vector OP from the origin of coordinates has length ρ and makes an angle θ with the X-axis. The two systems of representation are related to each other by the following formulas:

(1) $x = \rho \cos \theta$, (3) $\tan \theta = \frac{y}{x}$ or $\theta = \arctan \frac{y}{x}$

(2) $y = \rho \sin \theta$, (4) $\rho = \sqrt{x^2 + y^2}$

Finally, the complex number $x + iy$ may be regarded as a vector given in terms of its components x and y and the complex operator $i = \sqrt{-1}$. In practical work the symbol j is preferred to i, to avoid confusion with the symbol often used for the *current* in electricity.

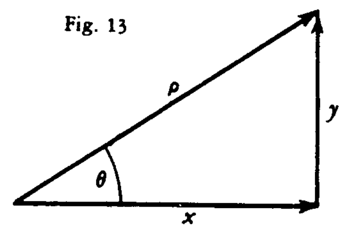
The "Euler identity" $e^{j\theta} = \cos \theta + j \sin \theta$ can be proved by use of the series expansions of the functions involved. Then $\rho e^{j\theta}$ is an *exponential* representation of the complex number $x + jy$, since $\rho e^{j\theta} = \rho \cos \theta + j\rho \sin \theta = x + jy$. The notation is often simplified by writing $\rho \underline{/\theta}$ in place of $\rho e^{j\theta}$.

If two or more complex numbers are to be added or subtracted, it is convenient to have them expressed in the form $x + jy$, since if $N_1 = x_1 + jy_1$, and $N_2 = x_2 + jy_2$, then $N_1 + N_2 = (x_1 + x_2) + j(y_1 + y_2)$. If, however, two or more complex numbers are to be multiplied, it is convenient to have them expressed in the exponential form. Then if $N_1 = \rho_1 e^{j\theta_1}$ and $N_2 = \rho_2 e^{j\theta_2}$, then $N_1N_2 = \rho_1\rho_2 e^{j(\theta_1+\theta_2)}$, or $(\rho_1 \underline{/\theta_1})(\rho_2 \underline{/\theta_2}) = \rho_1\rho_2 \underline{/\theta_1 + \theta_2}$.

It is therefore necessary to be able to change readily from either of these representations of a complex number to the other.

Changing from Components to Exponential Form

Fig. 13



If a complex number $x + jy$ (or vector in terms of perpendicular components) is given, the problem of changing to the form $\rho \underline{/\theta}$ is equivalent to finding the hypotenuse and one acute angle of a right triangle. The formulas $\tan \theta = \frac{y}{x}$ and $\rho = y/\sin \theta$, or $\rho = x/\cos \theta$, are the basis of the solution. Thus if $N = 4 + j3$, when 4 of C is set opposite 3 of D, the value of the ratio $\frac{y}{x}$, or $\frac{3}{4} = .75$ is read on D under the C index.

If the indicator is set at the index, and the slide moved so that .75 is under the hairline, the value of $\theta = 36.9°$ may be read on the T scale. Then $\rho = 3/\sin 36.9$ may be computed by moving the indicator to 3 on the D scale, pulling 36.9 on the S scale under the hairline, and reading $\rho = 5$ on the D scale opposite the left index of C. However, this method involves several unnecessary settings and is thus more subject to error than the method given in the general rule below.

A single T scale reading from 5.7° to 45° (left to right) and from 45° to 84.3° (right to left) will enable you to make all calculations. In order to provide ease in reading and to simplify the solution of certain problems the T scale on some models of slide rules is doubled. That is, the scale for tangents of angles from 5.7° to 45° is above the line and the scale for angles from 45° to 84.3° is below the line. Check your slide rule and determine the section of the manual that is applicable.

For single T scale

Observe that if x and y are both positive and $x = y$, then $\tan \theta = 1$ and $\theta = 45°$. If $y < x$, then $\theta < 45°$; if $y > x$ then $\theta > 45°$. Thus if $y < x$, the

T scale is read from *left* to *right*. If $y > x$, the T scale is read from *right* to *left*.

***Rule:* (i) To the *larger* of the two numbers (x, y) on D set an index of the slide. Set the indicator over the smaller value on D and read θ on the T scale. If $y < x$, then $\theta < 45$. If $y > x$, then $\theta > 45$, and is read from right to left (or on the left of the graduation mark).**

(ii) Move the slide until θ on scale S is under the indicator, reading S on the same side of the graduation as in (i). Read ρ on D at the index of the C-scale.

Observe that the reading both begins and ends at an index of the slide. By this method the value of the ratio y/x occurs on the C (or CI) scale of the *slide* over the smaller of the two numbers, and the angle may be read immediately on the T scale without moving the slide. In using any method or rule, it is wise to keep a mental picture of the right triangle in mind in order to know whether to read θ on the T or on the ST scale. Thus if the ratio y/x is a small number, the angle θ is a small angle, and must be read on the ST scale. To be precise, if $y/x < 0.1$, the ST scale must be used. Similarly, if the ratio $y/x > 10$, the angle θ will be larger than 84.3° and cannot be read on the T scale. The complementary angle $\varphi = (90 - \theta)$ will, however, then be on the ST scale, and then θ may be found by subtracting the reading on the ST scale from 90°, since $\theta = 90 - \varphi$.

EXAMPLES:

(a) Change $2 + j3.46$ to exponential or "vector" form. Note $\theta > 45$, since $y > x$ (or $3.46 > 2$). Set right index of S opposite 3.46 on D. Move indicator to 2 on D. Read $\theta = 60°$ on T at the *left* of the hairline. Move slide until 60° on scale S is under the hairline (numerals on the *left*), and read $\rho = 4$ on the D scale at the C-index. Then $2 + j3.46 = 4\rho e^{j60} = 4\ \underline{/60°}$.

(b) Change $3 + j2$ to exponential or vector form. Note that $\theta < 45°$ since $y < x$ (second component less than first). Set right index of S over 3 on D. Move indicator to 2 on D, read $\theta = 33.7°$ on T (use numerals on the right-hand side of graduations). Move slide until 33.7° on Scale S is under the hairline (numerals on right), and read 3.60 on the D scale at the C index. Hence $3+j2=3.60\ \underline{/33.7}$.

(c) Change $2.34 + j.14$ to exponential form. Since $y < x$, then $\theta < 45°$. Moreover, the ratio y/x is a small number (actually about .06). Since the tangent has one zero, the angle may be read on the ST scale. Set right index of S opposite 2.34 of D. Move indicator to .14 on D. Read $\theta = 3.43°$ on ST. The slide need not be moved. The value of ρ is approximately 2.34. In other words, the angle is so small that the hypotenuse is approximately equal to the longer side. Then $2.34 + j.14 = 2.34\ \underline{/3.43}$.

(d) Change $1.08 + j26.5$ to exponential form. Here $y > x$, so that $\theta > 45°$. But $\dfrac{y}{x} = \dfrac{26.5}{1.08} > 10$. Set right index of S on 26.5 of D. Move indicator to 1.08 of D. Read $\varphi = 2.34°$ on ST. The slide need not be moved. The value of ρ is approximately 26.5; $\theta = 90 - 2.34° = 87.66°$. Hence $26.5\ \underline{/87.66°}$ is the required form.

The following method of changing $x + jy$ to the form $\rho\ \underline{/\theta}$ using the DI scale is sometimes easier to use than methods based on the D scale.

Rule: (i) To the *smaller* of the two numbers (x, y) on DI set an index

of the slide. Set the indicator over the larger value on DI and read θ on the T scale. If $y < x$, then $\theta < 45°$. If $y > x$, then $\theta > 45°$ and is read from right to left (or on the left of the graduation mark).

(ii) Move the indicator over θ on scale S (or ST), reading S on the same side of the graduation as in (i). Read ρ on DI under the hairline.

EXAMPLES:

(a) Change $2 + j\,3.46$ to exponential form. Note that $y > x$ since $3.46 > 2$, and hence $\theta > 45°$. Set right index of C over 2 on DI. Move indicator to 3.46 on DI. Read $\theta = 60°$ on T. Move indicator to 60° on S. Read $\rho = 4$ on DI. Hence $2 + j\,3.46 = 4\underline{/60°}$.

(b) Change $114 + j\,20$ to exponential form. Here $y < x$, so $\theta < 45°$. Set left index of C over 20 on DI. Move indicator to 114 on DI. Read $\theta = 9.95°$ on T. Move hairline to 9.95° on S. Read $\rho = 116$ on DI. Hence $114 + j\,20 = 116\ \underline{/9.95°}$.

It will be observed that this rule is, in general, easy to use. In step (i) the value of tan θ for $\theta < 45°$ may be observed under the hairline on the C scale, and the value of tan θ for $\theta > 45°$ under the hairline on CI.

It may be noted that the rule given first (using the D scale) obtains the result in example (b) above without having the slide project far to th right. Thus, it appears that the relative advantages of the two methoc depend in part upon the problem.

If x and y are both positive, $\theta < 90°$. If x and y are *not* both positive, the resultant vector does not lie in the first quadrant, and θ is not an acute angle. In using the slide rule, however, x and y must be treated as both positive. It is therefore necessary to correct θ as is done in trigonometry when an angle is not in the first quadrant.

EXAMPLES:

(a) Find the angle between the X-axis and the radius vector for the complex number $-4 + j3$. First solve the problem as though both components were positive. The angle θ obtained is 36.9°. In this case the required angle is $180° - \theta = 180° - 36.9° = 143.1°$. Hence $-4 + j3 = 5\ \underline{/143.1°}$.

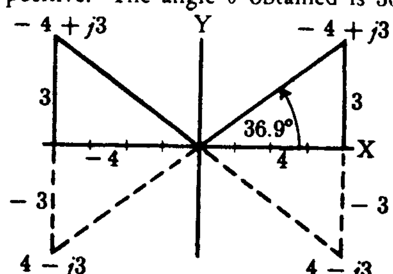


Fig. 14

Similarly for $-4 - j3$, the required angle is $180 + \theta = 180 + 36.9° = 216.9°$, so $-4 - j3 = 5\ \underline{/216.9°}$.

For $4 - j3$ the required angle is $360° - \theta = 323.1°$, so $4 - j3 = 5\ \underline{/323.1°}$, which may also be expressed in terms of a negative angle as $5\ \underline{/-36.9°}$.

(b) Change $17.2 - j6.54$ to exponential form. Here the ratio y/x is negative so θ can be expressed as a negative angle. In numerical value $y < x$, so the numerical or absolute value of $\theta < 45°$. Set left index of S opposite 17.2 on D. Move indicator over 6.54 of D, read $\theta = 20.8°$ on T. Pull 20.8 of S under hairline, read 18.4 on D at left index. Hence $17.2 - j6.54 = 18.4\ \underline{/-20.8°}$, or $18.4\ \underline{/339.2°}$.

For double T scale

Observe that if x and y are both positive and $x = y$, then $\tan\theta = 1$ and $\theta = 45°$. If $y < x$, then $\theta < 45°$; if $y > x$ then $\theta > 45°$. Thus if y $< x$, the upper T scale is used. If $y > x$, the lower T scale is used.

Rule. **(i) to x of the two numbers (x, y) on D set an index of the slide. Set the indicator over the value of y on D and read θ on the T scale. If y $<$ x, then $\theta < 45°$, and is found on the upper T scale. If y $>$ x, then $\theta > 45$, and is read on the lower T scale.**
(ii) Move the slide until θ on the S scale is under the hairline. Interchange the indices of the C scale if necessary. Read ρ on D under the index of the C scale.

Observe that the reading both begins and ends at an index of the slide. By this method the value of the ratio y/x occurs on the C (or CI) scale of the *slide* over y of the two numbers, and the angle may be read immediately on the T scale without moving the slide. In using any method or rule, it is wise to keep a mental picture of the right triangle in mind in order to know whether to read θ on the T or on the ST scale. Thus if the ratio y/x is a small number, the angle θ is a small angle, and must be read on the ST scale. To be precise, if $y/x < 0.1$, the ST scale must be used. Similarly, if the ratio $y/x > 10$, the angle θ will be larger than 84.3° and cannot be read on the T scale. The complementary angle $\varphi = (90 - \theta)$ will, however, then be on the ST scale, and then θ may be found by subtracting the reading on the ST scale from 90°, since $\theta = 90 - \varphi$.

EXAMPLES:

(a) Change $2 + j3.46$ to exponential or "vector" form. Note $\theta > 45$, since $y<x$ (or $3.46 > 2$). Set left index of S opposite 2 on D. Move indicator to 3.46 on D. Read $\theta = 60°$ on lower T. Move slide until 60° on scale S is under the hairline (numerals on the right), and read $\rho = 4$ on the D scale at the C-index. Then $2 + j3.46 = 4_{\epsilon}\, j60 = 4/\underline{60°}$.

(b) Change $3 + j2$ to exponential or vector form. Note that $\theta < 45°$ since $y < x$ (second component less than first). Set right index of S over 3 on D. Move indicator to 2 on D, read $\theta = 33.7°$ on upper T. Move slide until 33.7° on scale S is under the hairline (numerals on right), and read $\rho = 3.60$ on the D scale at the C index. Hence $3 + j2 = 3.60\ /\underline{33.7.}$

(c) Change $2.34 + j.14$ to exponential form. Since $y < x$, then $\theta < 45°$. Moreover, the ratio y/x is a small number (actually about .06). Since the tangent has one zero, the angle may be read on the ST scale. Set right index of S opposite 2.34 of D. Move indicator to .14 on D. Read $\theta = 3.43°$ on ST. The slide need not be moved. The value of ρ is approximately 2.34. In other words, the angle is so small that the hypotenuse is approximately equal to the longer side. Then $2.34 + j.14 = 2.34\ /\underline{3.43}$.

(d) Change $1.08 + j26.5$ to exponential form. Here $y > x$, so that $\theta > 45°$. But $\frac{y}{x} = \frac{26.5}{1.08} > 10$. Set right index of S on 26.5 of D. Move indicator to 1.08 of D. Read $\varphi = 2.34°$ on ST. The slide need not be moved. The value of ρ is approximately 26.5; $\theta = 90 - 2.34° = 87.66°$. Hence $26.5\ /\underline{87.66°}$ is the required form.

The following method of changing $x + jy$ to the form $\rho\ /\underline{\theta}$ using the DI scale is sometimes easier to use than methods based on the D scale.

Rule: **(i) To y of the two numbers (x, y) on DI set an index of the slide. Set the indicator over the value of x on DI and read θ on the T scale. If $y < x$, then $\theta < 45°$, and is found on the upper T scale. If $y > x$, then $\theta > 45°$ and is read on the lower T scale.**

(ii) Move the indicator over θ on scale S (or ST). Interchange the indices of the C scale if necessary. Read ρ on DI under the hairline.

EXAMPLES:

(a) Change $2 + j3.46$ to exponential form. Note that $y>x$ since $3.46 > 2$, and hence $\theta > 45°$. Set left index of C over 3.46 on DI. Move indicator to 2 on DI. Read $\theta = 60°$ on lower T. Interchange index of C scale. Move indicator to 60° on S. Read $\rho = 4$ on DI. Hence $2 + j3.46 = 4\underline{/60°}$.

(b) Change $114 + j20$ to exponential form. Here $y < x$, so $\theta < 45°$. Set the left index of C over 20 on DI. Move indicator to 114 on DI. Read $\theta = 9.95°$ on T. Move hairline to 9.95° on S. Read $\rho = 116$ on DI. Hence $114 + j20 = 116\ /\underline{9.95}°$.

It will be observed that this rule is, in general, easy to use. In step (i) the value of $\tan\theta$ may be observed under the hairline on the C scale. It may be noted that the rule given first (using the D scale) obtains the result in example (b) above without having the slide project far to the right. Thus, it appears that the relative advantages of the two methods depend in part upon the problem.

If x and y are both positive, $\theta < 90°$. If x and y are *not* both positive, the resultant vector does not lie in the first quadrant, and θ is not an acute angle. In using the slide rule, however, x and y must be treated as both positive. It is therefore necessary to correct θ as is done in trigonometry when an angle is not in the first quadrant.

EXAMPLES:

(a) Find the angle between the X-axis and the radius vector for the complex number $-4 + j3$. First solve the problem as though both components were positive. The angle θ obtained is 36.9°. In this case the required angle is $180° - \theta = 180° - 36.9° = 143.1°$. Hence $-4 + j3 = 5\ /\underline{143.1°}$. Similarly for $-4 - j3$, the required angle is $180 + \theta = 180 + 36.9° = 216.9°$, so $-4 - j3 = 5\ /\underline{216.9°}$. For $4 - j3$ the required angle is $360° - \theta = 323.1°$, so $4 - j3 = 5\ /\underline{323.1°}$, which may also be expressed in terms of a negative angle as $5\ /\underline{-36.9°}$.

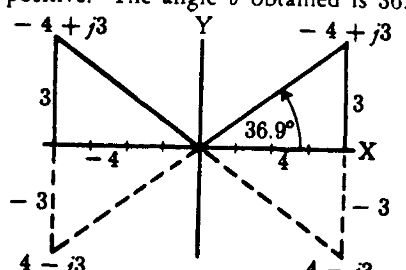


Fig. 14

(b) Change $17.2 - j6.54$ to exponential form. Here the ratio y/x is negative so θ can be expressed as a negative angle. In numerical value $y < x$, so the numerical or absolute value of $\theta < 45^\circ$. Set left index of S opposite 17.2 on D. Move indicator over 6.54 of D, read $\theta = 20.8^\circ$ on T. Pull 20.8 of S under hairline, read 18.4 on D at left index. Hence $17.2 - j6.54 = 18.4\ \underline{/-20.8^\circ}$, or $18.4\ \underline{/339.2^\circ}$.

Changing from Exponential Form to Components

The process of changing a complex number or vector from the form $\rho e^{j\theta} = \rho\ \underline{/\theta}$ to the form $x + jy$ depends upon the formulas $x = \rho \cos \theta$, $y = \rho \sin \theta$. These are simple multiplications using the C, D, and S (or ST) scales.

Rule: **Set an index of the S scale opposite ρ on the D scale. Move indicator to θ on the S (or ST) scale, reading from left to right (sines). Read y on the D scale. Moving indicator to θ on the S (or ST) scale, reading from right to left (cosines), read x on the D scale.**

If $\theta > 90^\circ$ or $\theta < 0$, it should first be converted to the first quadrant, and the proper negative signs must later be associated with x or y.

EXAMPLES:

(a) Change $4\ \underline{/60^\circ}$ to component form. Set right index of S on 4 of D. Move indicator to 60° on S (reading scale from left to right). Read 3.46 on D under hairline. Move indicator to 60° on S, reading scale from right to left (cosines). Read 2 on D under hairline. Hence $4\ \underline{/60^\circ} = 2 + j3.46$.

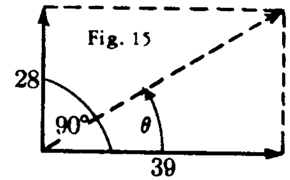
(b) Change $16.3\ \underline{/15.4^\circ}$ to the $x + jy$ form. Set left index of S on 16.3 of D. Move indicator to 15.4°of S, read 4.33 on D. Since 15.4° reading from right to left is off the D scale, exchange indices so the right index of C is opposite 16.3 of D. Move indicator to 15.4 of S, and read 15.7 on D. Hence $16.3\ \underline{/15.4^\circ} = 15.7 + j4.33$.

(c) Change $7.91\ \underline{/3.25^\circ}$ to component form. Set right index of S on 7.91 of D. Move indicator to 3.25 on ST. Read 0.448 on D. To determine the decimal point, observe that the angle is small, and hence the y component will also be small. Obviously, when the hypotenuse is near 8, 4.48 would be too large, and 0.0448 too small, to produce an angle of 3.25°. The cosine cannot be set on ST, but the angle is so small that the x-component is practically equal to the radius vector or hypotenuse. Hence 7.90 is a close approximation, and $7.91\ \underline{/3.25^\circ} = 7.90 + j0.448$.

(d) Convert $263\ \underline{/160^\circ}$ to the $x + jy$ form. Since $160^\circ > 90^\circ$, compute $180^\circ - 160^\circ = 20^\circ$. Set left index of the S scale on 263 of D. Move indicator to 20° on S. Read 90.0 on D. Move the slide so that the right index of S is on 263 of D. Move indicator to 20 (reading from right to left) on S. Read 247 on D. Since the angle is in the second quadrant, $263\ \underline{/160^\circ} = -247 + j90$.

ILLUSTRATIVE APPLIED PROBLEMS

1. Two forces of magnitude 28 units and 39 units act on the same body but at right angles to each other. Find the magnitude and angle of the resultant force.



In complex number notation, the resultant is $39 + j28$. Change this to exponential form. Since $28 < 39$, then $\theta < 45^\circ$. Set the right index of S on 39 of D. Move indicator to 28 of D. Read $\theta = 35.6^\circ$ on T. Move slide so 35.6° on S is under the hairline. Read $\rho = 48.0$ on D under the S-index. Hence the resultant has magnitude 48 units, and acts in a direction 35.6° from the larger force and $90 - 35.6^\circ$ or 54.4° from the smaller force. This angle can be read on the S scale under the hairline

2. A certain alternating generator has three windings on its armature. In each winding the induced voltage is 266.4 volts effective. The windings are connected in such a way that the voltages in each are given by the following vector expressions.

$$E_1 = 266.4\ (\cos 0^\circ - j \sin 0^\circ)$$

$$E_2 = 266.4\ (\cos 120^\circ - j \sin 120^\circ)$$
$$= 266.4 \cos 120^\circ - j\, 266.4 \sin 120^\circ$$

$$E_3 = 266.4\ (\cos 240^\circ - j \sin 240^\circ)$$
$$= 266.4 \cos 240^\circ - j\, 266.4 \sin 240^\circ$$

Express these numerically.

$$E_1 = 266.4\ (1 - j0) = 266.4 - j0$$

To find E_2, reduce the angles to first quadrant by taking $180^\circ - 120^\circ = 60^\circ$. Set the right index of S on 266.4 of D. Move the indicator to 60° of S (reading right to left). Read 133.2 on D. Move indicator to 60° on S, read 230.7 on D. Then

$$E_2 = -133.2 - j230.7$$

To find E_3, reduce 240° to the first quadrant by noting $240^\circ = 180^\circ + 60^\circ$. Hence, except for a negative sign, E_3 is the same as E_2, and

$$E_3 = -133.2 + j230.7$$

Suppose the first and second windings are so connected that their voltages subtract; that is,

$$E_0 = E_1 - E_2 = (266.4 - j0) - (-133.2 - j230.7) = 399.6 + j230.7$$

This may be changed to the $\rho\ \underline{/\theta}$ form. Set the right index of S on 399.6 of D. Move the indicator to 230.7 of D. Read $\theta = 30^\circ$ on T. Move slide so that 30° on S is under indicator, and read 461 on D at the S-index. Then $E_0 = 461\ \underline{/30^\circ}$, and hence the voltage is 461 volts and leads the voltage E_1 by 30°.

3. An alternating voltage of $104 + j60$ is impressed on a circuit such that the resulting current is $24 - j32$. Find the power and power factor. First convert each vector to exponential form.

$$E = 104 + j60 = 120 \underline{/30^\circ} \text{ volts, approximately}$$
$$I = 24 - j32 = 40 \underline{/-53.1} \text{ amperes, approximately.}$$

Hence the voltage leads the current by $30^\circ - (-53.1) = 83.1^\circ$.
The power factor $\cos 83.1^\circ = 0.120$.
The power $P = EI \cos \theta = (120)(40)(0.120) = 576$ watts, approximately.

4. The "characteristic impedance" of a section of a certain type of line is given by the formula $Z_0 = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}$, where in each case, the symbol Z represents a vector quantity. Compute Z_0 when

$$Z_1 = 40 + j120,\ Z_2 = 220 - j110.$$

First convert to exponential form.

$$Z_1 = 40 + j120 = 126 \underline{/71.6^\circ}$$
$$Z_2 = 220 - j110 = 246 \underline{/-26.6^\circ}$$

Hence

$$Z_1Z_2 = (126)(246) \underline{/71.6 - 26.6}$$
$$= 31{,}000 \underline{/45.0^\circ}$$
$$\frac{Z_1^2}{4} = \frac{126^2}{4} \underline{/2(71.6)}$$
$$= \frac{15{,}900}{4} \underline{/143.2}$$
$$= 3{,}975 \underline{/143.2}$$
$$Z = \sqrt{31{,}000 \underline{/45.0^\circ} + 3{,}975 \underline{/143.2}}$$

Since vectors are to be added before the square root is found, it is now convenient to convert them to component form.

$$31{,}000 \underline{/45.0^\circ} = 21{,}900 + j21{,}900$$
$$3{,}975 \underline{/143^\circ} = -3{,}180 + j2{,}390$$

To compute the latter, take $180^\circ - 143^\circ = 37^\circ$, compute the components using 37°, and observe that the x or real component must be negative since 143° is an angle in the second quadrant. Then

$$Z = \sqrt{(21{,}900 - 3180) + j(21{,}900 + 2390)}$$
$$= \sqrt{18{,}720 + j24{,}290}$$

In order to find the square root, it is convenient to change back to exponential form.

$$Z = \sqrt{18{,}720 + j24{,}290} = \sqrt{30{,}600 \underline{/52.4^\circ}}$$
$$= 175 \underline{/26.2^\circ} \text{ ohms.}$$

The final result is obtained by setting 30,600 on D and reading 175 on $\sqrt{\ }$; the angle 52.4° is merely divided by 2. This problem shows the value of being able to change readily from one form of vector representation to the other.

PART 4. USE OF LOG LOG SCALES

To find the value of 1.3^7, $5.6^{3.21}$, $\sqrt[5]{38}$, $\sqrt[3.5]{84}$, and many other types of expressions, Log Log scales are used. The method of computing such expressions will be explained in later sections. First the Log Log scales will be described.

The Log Log scale has two main parts. One part is used for numbers greater than 1. The other part is used for numbers between 0 and 1; that is, for proper fractions expressed in decimal form. On some models of the slide rule these two parts are arranged "back to back." One part, indicated by LL1+, is above the line. The other part, indicated by LL1− is under the line. (See Fig. 16). On other models the parts are separated.

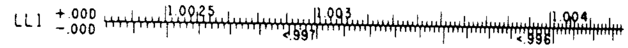


Fig. 16

READING THE SCALES

Numbers greater than 1:

On an ordinary logarithmic scale, such as the D scale, any particular graduation represents many different numbers. Thus the graduation labeled 2 represents not only 2, but also 20, 200, .2, .02, etc. In contrast, any graduation on a Log Log scale represents only one number. The principal graduations are labeled with a number in which the decimal point is shown.

The scales labeled LL1+, LL2+, LL3+, LL4+ are sections of one continuous scale about 40 inches long. The top scale, marked LL1+, begins at the left end at about 1.00230. Set the hairline of the indicator on this mark, then move the indicator slowly to the right, reading 1.0025, 1.003, etc., ending at 1.0232. When the end of the scale is reached, move the indicator to the left end of the rule and continue reading on the LL2+ scale, reading 1.03, 1.04, etc., to about 1.259. The scale marked LL3+ begins at 1.259 and ends at 10. Finally, the scale LL4+ begins at 10 and ends at 10^{10} or 10,000,000,000 (ten billion).

There is no difficulty in reading the principal graduations since they are labeled and the decimal point is shown. Between the principal graduations the intervals are subdivided in several different ways. Thus the graduations between the numbers shown do not have the same meaning in all sections of the scale. To the beginner, this variation in the meaning of the scale divisions is often confusing. However, as one gains familiarity with the instrument, the proper reading usually may be obtained at a glance. The basic scheme is the same as that used in sub-dividing ordinary logarithmic scales, such as C and D.

(1) To locate a number, look first for the nearest smaller number that appears on the scale.

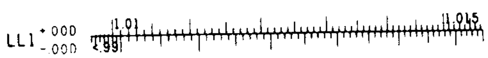
(2) Second, observe the major subdivisions between the nearest smaller number and the one following it. Sometimes there are 10, at other times 5, and at still other times only 2 or 3 major parts of the interval.

The general idea used in reading the scales may be stated informally as follows: Starting with the smaller printed number, decide how you must "count" the major graduation marks to come out correctly at the larger printed number.

(3) Third, in most cases there are still other or minor subdivisions between the major ones. These minor subdivisions divide the major intervals into 10 sub-parts, 5 sub-parts, or 2 sub-parts. Use the slide rule and check the location of the numbers in the following table.

Number		Scale
1.01278	is between 1.01 and 1.015 on	LL1 +
1.173	is between 1.15 and 1.2 on	LL2 +
4.78	is between 4 and 5 on	LL3 +
1.054	is between 1.05 and 1.06 on	LL2 +
1.862	is between 1.8 and 1.9 on	LL3 +
25.6	is between 20 and 30 on	LL4 +

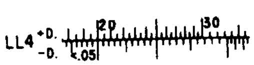
Examples: *Count by "ones" (thousandths)*
(a) *Then by "ones" again (tenths of thousandths)*



5 major sub-parts
10 minor sub-parts

In this example, there are 10 minor sub-parts. Each represents one-tenth of the interval. Starting at 1.01, the graduations read in sequence represent the following numbers: 1.0101, 1.0102, 1.0103, 1.0104, 1.0105, 1.0106, 1.0107, 1.0108, 1.0109, 1.0110, 1.0111, etc.

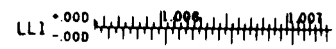
(b) ***Count by "ones" (units)***
Then by "fives" (halves of units)



10 major sub-parts
2 minor sub-parts

In this example the minor subdivisions represent halves of the major interval. Hence the graduations read in order represent 20, 20.5, 21, 21.5, etc.

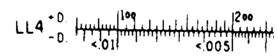
(c) ***Count by "ones" (tenths of thousandths)***
Then by "fives" (hundredths of thousandths)



10 major sub-parts
2 minor sub-parts

In this example the minor subdivisions represent half the interval, and the readings in order are 1.006, 1.0065, 1.00605, 1.0061, 1.00615, etc.

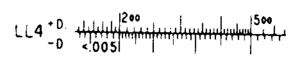
(d) ***Count by "tens" (ten)***
Then by "fives" (five)



10 major sub-parts
2 minor sub-parts

In this example the minor subdivisions represent halves of the major sub-interval. The readings in order are 100, 105, 110, 115, etc.

(e) ***Count by "hundreds" (hundred)***
Then by "tens" (ten)



3 major sub-parts
10 minor sub-parts

In this example the minor subdivisions represent tenths of the major sub-intervals. The graduations read in order represent 200, 210, 220, 230, etc.

The general idea is the same in all cases. It is necessary to decide how the marks must be "counted" to come out right. That is, if the "counting" is properly done, it "comes out right" when the next principal graduation (labelled with a number) is reached.

Numbers from 0 to 1:

The scales labeled LL1–, LL2–, LL3–, LL4– are parts of one continuous scale 40″ long reading from about .9977 to 10^{-10} or .000,000,0001. The ranges of these sections are approximately as follows.

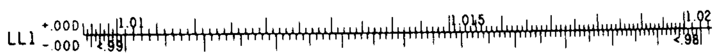
SCALE	LEFT INDEX	RIGHT INDEX
LL1 –	.9977	.977
LL2 –	.977	.794
LL3 –	.794	.100
LL4 –	.100	.000,000,000,1

The methods of subdividing these scales are the same as those used for numbers greater than 1. The methods of reading the scales are also the same. Use the slide rule to check the location of the numbers in the table below. As before, look first for the nearest smaller number at a principal graduation mark.

NUMBER	SCALE
.984 between .98 and .99 on	LL1–
.813 between .80 and .85 on	LL2–
.231 between .20 and .25 on	LL3–
.026 between .01 and .05 on	LL4–

The examples below show how the scales may be read.

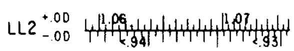
(f) *Count by "ones" (thousandths)*
Count by "ones" (tenths of thousandths)



10 major sub-parts
10 minor sub-parts

In this example there are 10 minor sub-parts. Each represents one-tenth of the interval. The graduations read in order from right to left represent .9801, .9802, .9803, .9804, .9805, etc.

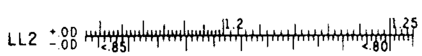
(g) *Count by "ones" (thousandths)*
Then by "fives" (tenths of thousandths)



10 major sub-parts
2 minor sub-parts

In this example there are 2 minor sub-parts. The graduations read in order from right to left represent .9300, .9305, .9310, .9315, .9320, etc.

(h) *Count by "ones" (hundredths)*
Then by "ones" (thousandths)



5 major sub-parts
10 minor sub-parts

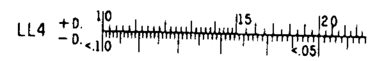
In this example there are 10 minor sub-parts. Each represents one-tenth of the interval. The graduations read in order from right to left represent 800, .801, .802, .803, .804, .805, .806, etc.

(i) ***Count by "ones" (hundredths)***
Then count by "twos" (thousandths)

5 major sub-parts
5 minor sub-parts

In this example there are 5 minor sub-parts. Each represents one-fifth of the interval. The graduations read in order from right to left represent .400, .402, .404, .406, .408, .410, etc.

(j) ***Count by "ones" (hundredths)***
Then by "ones" (thousandths)



5 major sub-parts
10 minor sub-parts

In this example there are 10 minor sub-parts. Each represents one tenth of the interval. The graduations read in order from right to left represent .050, .051, .052, .053, .054, .055, etc.

FUNDAMENTAL RELATIONSHIPS

There is a reciprocal relationship between numbers set on the Log Log scales. This relationship plays an important role in the use of these models of the slide rule.

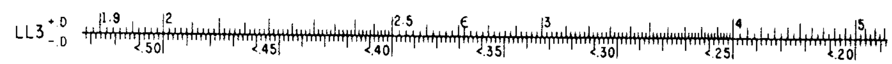


Fig. 17

Check the readings in the table below on the slide rule. Observe that the symbols for the scales have been chosen to emphasize this relationship.

Ex.	Number	Scale	Reciprocal	Scale
(a)	2	LL3+	.5	LL3–
(b)	5	LL3+	.20	LL3–
(c)	1.25	LL2+	.80	LL2–
(d)	1.0131	LL1+	.9871	LL1–
(e)	52	LL4+	.0192	LL4–

It should be recalled that, in general, a number *N* may be represented by the form b^m. In this form the number *b* is called the *base* and the number *m* is called the *exponent*. The number *N* is called the *power*. In this discussion the number *b* will always be greater than o and not equal to 1, (i.e., $b>o$ and $b \neq 1$).

By definition, the logarithm of a number N to the base b is the exponent that must be given to b to produce N. The number m in the expression above is also the logarithm.

Exponential Form	*Logarithmic Form*
$N=b^m$	$\log_b N=m$

Although the Log Log scales of a slide rule have important uses in connection with the exponential form, it will be convenient to consider first their use in finding logarithms.

In general, if N represents a number under the indicator hairline on a Log Log scale, the logarithm of N will be under the hairline of an ordinary Log scale, the logarithm of N will be under the hairline of an ordinary logarithmic scale, such as the D scale. The choice of the base b and the appropriate ordinary logarithmic scale is, however, affected by the type of scale arrangement available on the slide rule.

In scientific work the most convenient base is often the number e (approximately 2.718). Slide rule scales are therefore arranged to favor this base. When the base is e the logarithms are called natural, hyperbolic, or Napierian. Common logarithms have the number 10 as the base.

FINDING LOGARITHMS, BASE *e*.

On these models two special scales (DF/M and CF/M) are provided. (See page 21). They are ordinary logarithmic scales which are "folded" on 2.3, approximately. When logarithms to base e are to be found the following rule applies.

Rule (a): Position. **When the indicator is set over any number N on a Log Log scale, the numerical value of the natural logarithm may be read under the hairline on the DF/M scale, and conversely.**

(b) Sign: **If the number is greater than 1 (set on LL1+ to LL4+) the logarithm is positive.**

If the number is less than 1 (set on LL1– to LL4–) the logarithm is negative.

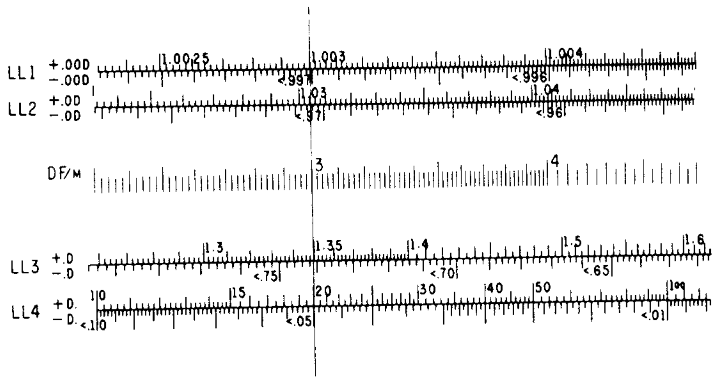


Fig. 18

(c) Decimal point. **Place the indicator hairline over 1 of DF/M. Note the following cases:**

(i) If N is on LL3+ and LL4+ between e and 22,000 (under e), the decimal point of the logarithm is at the right of the first digit read on DF/M. (Observe the reminder symbol D. at the left end of LL4+.) This symbol also governs the part of LL3+ to the right of e.

(ii) If N is on LL2+ and LL3+ between 1.105 (above e) and ***e***, the decimal point of the logarithm is moved one place to the left of the first digit read on DF/M. (Observe the reminder symbol .D at the left end of LL3+. This symbol also governs the part of LL2+ to the right of 1 on DF/M.)

(iii) If N is on LL2+ and LL1+ between 1.105 and 1.01, the decimal point of the logarithm is moved two places to the left of the first digit read on DF/M. (Observe the reminder symbol .OD at the left end of LL2+.) This symbol also covers the part of LL1+ to the right of 1 on DF/M.

(iv) If N is on LL1+ to the left of 1.01, the decimal point of the logarithm is moved three places to the left of the first digit read on DF/M. (Observe the reminder symbol .OOD at the left of LL1+.)

(v) If N is on LL4+ to the right of 1 on DF/M, the decimal point in the logarithm is moved two places to the right of the first digit read on DF/M. (Note: $\log_e 10^{10} = 10 \log_e 10 = 10\ (2.3) = 23$. This is the logarithm of the largest value on the Log Log scale.)

(i) to *(v)*. The same rules hold for number set on the scales LL4+ to LL1+.

For small values of x, it is true that $\log_e (1+x).=x$, approximately. Hence the logarithm of numbers on LL1+ is approximately equal to the decimal fraction following the 1 in N. Thus $\log_e (1.008) = .008$, approximately. If this fact is kept in mind, it is easy to place the decimal point in the logarithm. *The decimal point moves 1 place to the right each time the 1 of DF/M is crossed from left to right.*

Check on the slide rule the readings shown in the table below:

Ex.	Number	Scale	$\log_e$ N
(a)	4	LL3+	1.386
(b)	1.15	LL2+	0.1398
(c)	1.02	LL1+	0.0198
(d)	30	LL4+	3.4
(e)	1.405	LL3+	0.34
(f)	1.0346	LL2+	0.034
(g)	0.05	LL4–	–3.00
(h)	0.63	LL3–	–0.462
(i)	0.946	LL2–	–0.0555
(j)	0.9964	LL1–	–0.0036

FINDING LOGARITHMS, ANY BASE

Logarithms to any base a may be found by the formula

$$\log_a N = (\log_e N) \div (\log_e a)$$

For common logarithms $a = 10$, and $\log_e 10 = 2.303$. Since

$$1 \div 2.303 = .4343,$$

the formula becomes

$$\log_{10} N = .4343 \log_e N.$$

The value of $\log_{10} N$ may be read directly on the D scale. The symbols D., .D, .OD, and .OOD at the left end of the Log Log scales show how to place the decimal point for any number set on the corresponding scale. With this scale arrangement, it is as easy, if not easier, to find logarithms to base 10 as to base e.

Rule (a): Position. When the indicator is set over any number N on a Log Log scale, the numerical value of the logarithm to base 10 may be read under the hairline on the D scale, and conversely.

(b) Sign: If the number is greater than 1 (set on LL1+ to LL4+), the logarithm is positive.

If the number is less than 1 (set on LL– to LL4–) the logarithm is negative.

Check on the slide rule the readings shown in the table below:

Ex.	Number	Scale	$\log_{10}$ N
(a)	4	LL3+	.602
(b)	1.15	LL2+	.0607
(c)	1.02	LL1+	.00860
(d)	30	LL4+	1.477
(e)	1.405	LL3+	.1477
(f)	1.0346	LL2+	.01477
(g)	0.05	LL4–	–1.301
(h)	0.63	LL3–	–.201
(i)	0.946	LL2–	–.0241
(j)	0.9964	LL1–	–.00157

FINDING POWERS OF e

Powers of e are easily found by using the Log Log scales. Since if $m = \log_e N$, then by definition $e^m = N$, the process is the reverse of finding the logarithm.

***Rule:* To find a power of e, set the indicator over the exponent on the DF/M scale, and read the corresponding value on the Log Log scale. The appropriate scale is found by using the rules for the decimal point in the logarithm.**

Verify the following examples by use of the slide rule.

Ex.	Problem	Scale for Exponent	Power	Scale for Power
(a)	e^{3}	DF/M	20.1	LL4+
(b)	$e^{0.25}$	DF/M	1.284	LL3+
(c)	$e^{0.081}$	DF/M	1.0844	LL2+
(d)	$e^{-2.0}$	DF/M	0.135	LL3–
(e)	$e^{-0.20}$	DF/M	0.819	LL2–
(f)	$e^{-0.020}$	DF/M	0.9802	LL1–
(g)	$e^{-0.002}$	—	0.998	1.000-.002
(h)	e^{5}	DF/M	148.	LL4+
(i)	$e^{8.35}$	DF/M	4230.	LL4+
(j)	$e^{-3.4}$	DF/M	0.0334	LL4–

FINDING POWERS OF ANY BASE

The Log Log scales may be used to find any power of any base. Since roots may be expressed by exponents that are fractions in decimal form, the Log Log scales may also be used to find any root of a positive number. These statements are, of course, subject to certain restrictions which are of minor importance in practical work.

The problem is to compute $N = b^m$ when b and m are known numbers. The general method is given by the following rule.

Rule: To find b^m, when $m > 0$ set the index of an ordinary logarithmic scale on the slide (C, CF,) opposite b on a Log Log scale. Move the indicator to m of the ordinary logarithmic scale, and read b^m under the hairline on the Log Log scale.

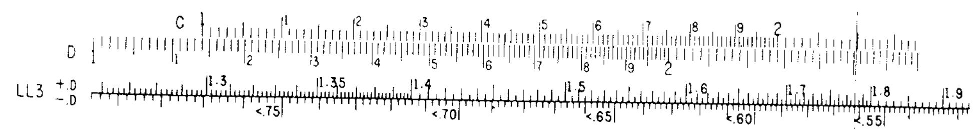


Fig. 19

Example: To find $N = 1.3^{2.2}$, set the index of the C scale over 1.3 of the Log Log scale. Move the indicator hairline over 2.2 of the C scale. Read 1.78 on the Log Log scale.

The rule is based on mathematical theory which may be illustrated as follows: Given $N = 1.3^{2.2}$, take logarithms of both sides. Then $\log_a N = 2.2 \log_a 1.3$.

To compute the right member, take logarithms of both sides again. Then

$$\log_{10} \log_a N = \log_{10} 2.2 + \log_{10} \log_a 1.3.$$

The graduation at 1.3 of a Log Log scale of the slide rule represents a length from the left index that is proportional to $\log_{10} \log_a 1.3$.

The graduation at 2.2 of the C scale represents a length that is proportional to $\log_{10} 2.2$. The lengths are added on the slide rule. The graduation on the Log Log scale at the sum represents N.

Since the *method* is simple, the main difficulty is to decide on which Log Log scale to read the result. The following principles will be helpful:

1) Remember that the Log Log scales for numbers greater than 1 are really sections of a single continuous scale. They could be arranged end-to-end on a long slide rule, with repeated lengths of D scale opposite them.

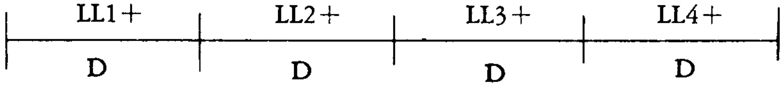


Fig. 20

Similarly, the Log Log scales for numbers less than 1 are really sections of a single scale. They also could be arranged end-to-end on a long slide rule.

2) Think of the scales arranged end-to-end as above. If the exponent m is greater than 1, then $N = b^m$ would be to the right of b. If the exponent m is positive but less than 1, then $N = b^m$ would be to the left of b.

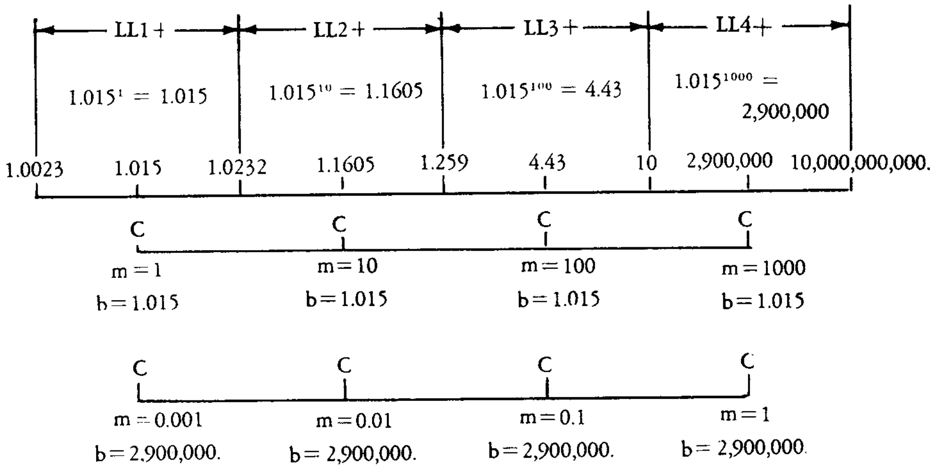


Fig. 21

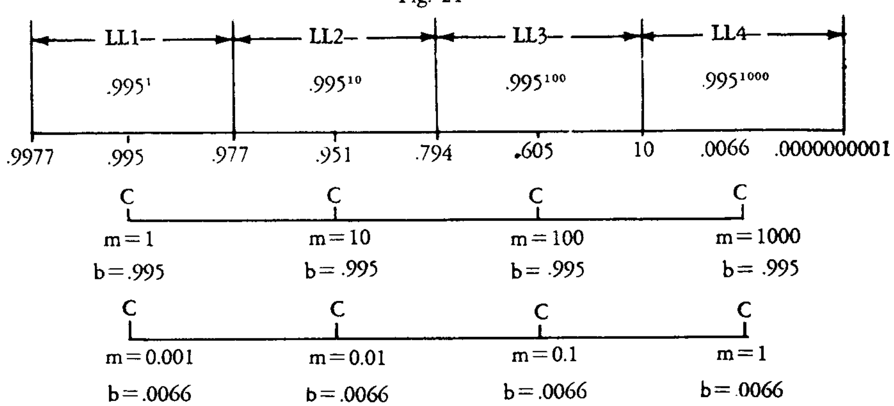


Fig. 22

3) Think of the scales arranged end-to-end as above. For any setting of the indicator hairline, the reading on any section of the Log Log scale is the 10th power of the reading on the adjacent section at its left. That is, *moving one scale section-length to the right has the effect of raising the number to the 10th power.* Conversely, moving one section-length to the left has the effect of taking the one-tenth power of the number.

Ex.	Number	Scale
(a)	$1.015^1 = 1.015$	LL1+
	$(1.015)^{10} = 1.1605$	LL2+
	$(1.015)^{100} = (1.1605)^{10} = 4.43$	LL3+
	$(1.015)^{1000} = (4.43)^{10} = 2{,}900{,}000.$	LL4+
(a')	$2{,}900{,}000^1 = 2{,}900{,}000.$	LL4+
	$(2{,}900{,}000)^{0.1} = 4.43$	LL3+
	$(2{,}900{,}000)^{0.01} = (4.43)^{0.1} = 1.1605$	LL2+
	$(2{,}900{,}000)^{0.001} = (1.1605)^{0.1} = 1.015$	LL1+
(b)	$0.995^1 = 0.995$	LL1–
	$(0.995)^{10} = 0.951$	LL2–
	$(0.995)^{100} = (0.951)^{10} = 0.605$	LL3–
	$(0.995)^{1000} = (0.951)^{100} = 0.0066$	LL4–
(b')	$0.0066^1 = 0.0066$	LL4–
	$(0.0066)^{0.1} = 0.605$	LL3–
	$(0.0066)^{0.01} = (0.605)^{0.1} = 0.951$	LL2–
	$(0.0066)^{0.001} = (0.951)^{0.1} = 0.995$	LL1–

Now $(b^m)^{10} = b^{10m}$. For example, $(4^{0.5})^{10} = 4^{10 \times 0.5} = 4^5$. Moving the decimal point in the exponent one place to the right is equivalent to raising the number to the 10th power, so that the power is on the adjacent scale to the right (or below). Moving the decimal point two places moves the number on the Log Log scale two sections, etc.

Moving the decimal point in the exponent one place to the *left* has the effect of moving the number on the Log Log scale one section to the *left*, (or above), etc.

Rule: Imagine the decimal point is at the right of the first significant digit of the exponent, and decide on which scale b^m would then be found. Then look to the "right" or "left" as many sections as are needed to adjust the decimal point to its true position.

Examples: (a) Find $(1.03)^{200}$. Set the left index of the C scale opposite (1.03) of the Log Log scale. Move the indicator over 2 on the C scale. Observe that $(1.03)^2 = 1.0609$ is on the same section, and that $(1.03)^{20} = 1.806$ is on the adjacent or next section of the Log Log scale. Finally, $(1.03)^{200} = 370$ is on the second section beyond $(1.03)^2$.

(b) Find $4^{0.05}$. Set the index of the C scale opposite 4 of the Log Log scale. Move the indicator over 5 on the C scale. Observe that 4^5 would be to the right of 4 at 1024, and $4^{0.5}$ is to the left of 4. In fact, $4^{0.5} = 4^{1/2} = \sqrt{4} = 2$. Then $4^{0.05}$ is one scale length farther to the left, and is 1.0718 on the next scale.

On some models a small chart (shown below) has been provided on the slide to help in deciding on which scale b^m is to be read. The left-hand part is read as follows. When the left index of the C scale is set over b of a Log Log scale, and there are 4 digits in m, then b^m is found 3 scales below (if there are that many on the rule). If there are 3 digits in m then b^m is found 2 scales below. If there is 1 digit in m, then b^m is found 0 scales below (i.e., on the same scale as b). If there are 2 zeros in m, then b^m is found 3 scales above the one on which b is located (if there are that many on the rule), etc.

L INDEX OF C ON b – INDICATOR ON m – R INDEX OF C ON b

D'S IN m OR Z'S IN m	OF C SCALE	DIGITS IN m OR Z'S IN m
4 3 2 1 0 1 2	b^m THEN FOUND	3 2 1 0 1 2 3
3 2 1 0	SCALES BELOW	3 2 1 0
0 1 2 3	SCALES ABOVE	0 1 2 3

Fig. 23

Note that if it is necessary to use the right index of an ordinary logarithmic scale opposite *b*, the value of b^m is read one scale to the *left* (or "below") where it would be if the left index could be used.

Examples:

Ex.	Problem $b^m = N$	Direction from b	Number of Sections from Scale of b	Answer N
(a)	$1.5^{2.4}$	Right	0	2.646
(b)	$1.5^{0.5}$	Left	1	1.225
(c)	1.0267^{32}	Right	1	2.32
(d)	$2.2^{0.3}$	Left	0	1.267
(e)	$2.2^{1.5}$	Right	0	3.26
(f)	$0.5^{1.2}$	Right	0	0.435
(g)	0.4^{2}	Right	0	0.16
(h)	$0.88^{0.25}$	Left	0	0.9685
(i)	$0.5^{0.05}$	Left	1	0.9659
(j)	$0.2^{0.008}$	Left	2	0.9872

Rule. When the exponent is negative, use the same procedure as for positive exponents, but read the final result on the ***reciprocal scale.***

Examples: (Compare with examples (a) to (j) above)

(a^1) Find $1.5^{-2.4}$. Compute $1.5^{2.4}$, but instead of reading the result as 2.646 on LL3+, read .378 on LL3– the reciprocal scale.

(b^1) $1.5^{-0.5}$. Compute $1.5^{0.5}$, but instead of reading the result as 1.225 on LL2+, read 0.8165 on LL2– the reciprocal scale. Fig. 24.

(f^1) Find $0.5^{-1.2}$. Compute $0.5^{1.2}$ as in example (f) above, but instead of reading 0.435 on LL3–, read the result 2.3 on LL3+ the reciprocal scale.

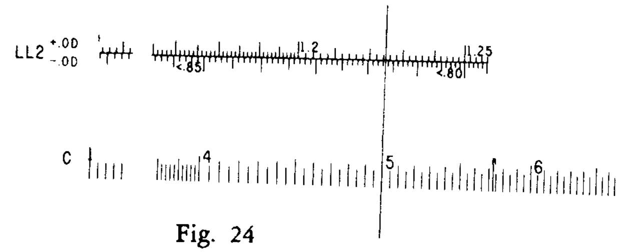


Fig. 24

PROBLEMS	ANSWERS
1. $4.3^{5.21}$	2000
2. $16.3^{0.107}$	1.348
3. $2.23^{0.073}$	1.060
4. $.325^{5.6}$	0.0018
5. $.734^{0.058}$	0.9822
6. $1.075^{-4.5}$	0.7223
7. $8.5^{-0.107}$	0.795

ROOTS, AND COMMON FRACTIONAL EXPONENTS

There are two methods of finding roots

Rule: To find $\sqrt[m]{b}$ or $b^{\frac{1}{m}}$ set the index of C scale on b on the Log Log scale, move indicator to m on CI, read result on Log Log scale under hairline.

This method uses the theory of exponents to express a root by using a fractional exponent (e.g. $=1/m$). This fraction can be divided out and the result used as an exponent as described under finding powers. Thus $\sqrt[4]{3} = (3)^{1/4} = 3^{.25}$. However, the CI scale does the division automatically, since it gives reciprocals of numbers on the C scale.

Moreover, in some applied problems the formulas being used express the exponent as a common fraction, and it is then more convenient to use the CI scale.

Examples:

(a) Find $\sqrt[4.2]{8.5}$ or $(8.5)^{\frac{1}{4.2}}$. Set right index of C scale over 8.5 on the Log Log scale. Move hairline to 4.2 on CI scale, read 1.664 on the Log Log scale under hairline.

(b) Find $\sqrt[.03]{0.964}$ or $(0.964)^{\frac{1}{.03}} = (0.964)^{33.3}$. Set the index of the C scale opposite 0.964 of the Log Log scale. Move the indicator over 3 on CI. Since $1/.03 = 33$, approximately, the result is one scale section to the right of the one on which 0.964 is set. Read the result as 0.295.

A second procedure treats roots as the inverse of powers.

Rule: To find $\sqrt[m]{b}$, or $b^{\frac{1}{m}}$ set hairline over *b* on a Log Log scale, pull *m* on the C scale under the hairline, read the result on the Log Log scale at the index.

Examples:

(a) Find $\sqrt[5]{6.3}$ or $(6.3)^{\frac{1}{5}}$. Set hairline over 6.3 on the Log Log scale, move slide so 5 of the C scale is under hairline, read 1.445 under left index on the Log Log scale.

(b) Find $\sqrt[4]{0.56}$ or $(0.56)^{\frac{1}{4}}$. Set hairline over 0.56 on the Log Log scale, move slide so 4 of C scale is under hairline, read 0.865 at right index of C on the Log Log scale.

The proper scale on which to read the root may be determined by reversing the methods used earlier for finding powers. By definition, $\sqrt[m]{b}$ is a number which raised to the *m* power produces *b*, that is $\left(\sqrt[m]{b}\right)^m=b$. Suppose $b>1$, and $m>1$; then $\sqrt[m]{b}<b$ and $\sqrt[m]{b}$ would be to the left of *b*, if the Log Log scales were on one continuous line. In example (a) above, $\sqrt[5]{6.3}$ or 1.445 is less than 6.3, and $1.445^5=6.3$. Although the reading on the LL2+ scale, or 1.0375, is also less than 6.3, it is the 50th root of 5.3, or $\sqrt[50]{6.3}$. On the other hand, the value of $\sqrt[0.5]{6.3}$ is to the right of 6.3 at about 40; observe that $\sqrt[0.5]{6.3}$ or $6.3^{1/.5}=6.3^2$ is about 40.

Examples:

(a) Find $y=\sqrt[400]{100}$. Set 4 of the C scale on 100 of the Log Log scale and move indicator to the C index. Note that the 4th root would be about 3, the 40th root about 1.12 and the 400th root must be 1.0116.

(b) Find $\sqrt[50]{0.05}$ or $0.05^{\frac{1}{50}}$. Set 5 on the C scale over 0.05 on the Log Log scale. At the index of the C scale read 0.9418 two scale sections to the left (above).

PROBLEMS	ANSWERS
1. $6.5^{\frac{1}{1.51}}$	3.45
2. $3400^{\frac{1}{75}}$	1.114
3. $1.606^{\frac{1}{21.5}}$	1.0223
4. $7.4^{\frac{53.6}{27.9}}$	47
5. $1.357^{\frac{4.21}{7.36}}$	1.191
6. $1.0411^{\frac{74.5}{9.8}}$	1.381

SOLVING EXPONENTIAL EQUATIONS:

The method of solving equations of the type $b^m=N$, where b and N are known and m is unknown, is very similar to the process of finding $b^m=N$ when m is known and N unknown. (See Finding Powers, above.)

Rule: Set the index of the C scale (or CF scale) on b. Move the hairline to N on a Log Log scale. Read m under the hairline on the C scale (or CF scale, if it was used).

Examples:

(a) Solve $1.37^m=8.43$. Set the index of the C scale opposite 1.37 of the Log Log scale. Move the hairline to 8.43 on the Log Log scale. Read 6.77 on the C scale under the hairline. It should be observed that 8.43 is greater than 1.37 and is to the right of 1.37 on the Log Log scale. Hence the exponent m must be larger than 1. The exponent 67.7 would obviously be too large, and it follows that the decimal point must be to the right of the 6 as in 6.77

(b) Solve $(0.75)^x=0.872$. Set the index of the C scale opposite 0.75 of the Log Log scale. Move the hairline to .872 on the Log Log scale. Read .476 on the C scale under the hairline. Observe that 0.872 is to the left of 0.75, so the exponent is less than 1. The number .872 is one scale above (to the left) of .75 so the decimal point must be at the left of the 4.

(c) Solve for y if $(0.94)^y=2.37$.

Fig. 25

Set the left index of the C scale opposite 0.94 on the Log Log scale. Move the indicator to 2.37 on the Log Log scale. Read 13.9 on the C scale. (The use of a folded C scale, such as CF, is convenient in this example.) Observe that in this example 0.94 is less than 1 and 2.37 is greater than 1. Hence the exponent y must be a negative number. The reciprocal of 0.94 is about 1.064, and 2.37 is one scale section to the right. Hence $y=-13.9$.

(d) Solve for *p* if $(5.27)^p=0.818$.

Set the index of the C scale at 5.27 of the Log Log scale. Move the indicator to 0.818 on the Log Log scale. Read 1209 on the C scale. Note that $5.27>1$ and $0.818<1$, so the exponent is a negative number. The reciprocal of 0.818 is about 1.222, and since this is less than 5.27 the numerical value of the exponent must be less than 1. Hence $p=-0.1209$.

It is useful to notice that when powers are being found, the logarithmic solution may often be directly observed on the C and D scales. As a simple example, consider finding $x=2^3$. Then $\log x=3\log 2$. When the left index of the C scale is set on 2 of the LL3+ scale, the logarithm of 2, or .301, is visible on the D scale under the index of the C scale. When the hairline is moved to 3 on the C scale, one may think of this operation as multiplying .301

by 3 by use of the C and D scales. The result is .903, read on the D scale, and this in turn is the logarithm of 8, read below it on the LL3+ scale.

PROBLEMS	ANSWERS
1. $4^x = 3.75$	$x = 0.953$
2. $.963^x = 0.823$	$x = 5.17$
3. $5.25^x = 1.0141$	$x = 0.00844$
4. $2.11^x = 11{,}000$	$x = 12.46$
5. $3.04^x = 0.85$	$x = -.146$
6. $1.475^x = 0.015$	$x = -10.8$

LOGARITHMS OF COMPLEX NUMBERS

The logarithm of a complex number $z = x + jy$ is a complex number. Let $\log_e (x + jy) = u + jv$. Then
$x + jy = e^{u+jv} = e^u . e^{jv} = e^u (\cos v + j \sin v) = e^u \cos v + j\, e^u \sin v$.
Equating the real and then the imaginary parts gives two equations

$$x = e^u \cos v$$
$$y = e^u \sin v$$

which may be solved for u and v. By division, $\tan v = y/x$, and hence $v = \arctan (y/x)$. Squaring and adding, $x^2 + y^2 = e^{2u}$, and hence $u = \log_e \sqrt{x^2 + y^2}$. Then

$$\log_e (x + jy) = \log_e \sqrt{x^2 + y^2} + j \arctan (y/x) = \log_e \rho + j\theta.$$

***Rule:* To find $\log_e (x + jy)$, first convert $x + jy$ to polar form $\rho\underline{/\theta}$. Find $\log_e \rho$ and write the results in the form $\log_e \rho + j\theta$.**

For double T scale

EXAMPLE: Find $\log_e (2.6 + j3.4)$. To convert $2.6 + 3.4$ to polar form, set left C-index over 2.6 on D. Move indicator to 3.4 on D. Read $\theta = 52.6°$ on lower T under hairline. Move slide to bring sin 52.6° on S (read left to right) under hairline, and find $\rho = 4.28$ on D. Set indicator on 4.28 of the LL3+ scale. Read 1.454 on the DF/M scale. Then $\log_e (2.6 + j3.4) = 1.454 + j52.6°$, or $1.454 + j0.92$, when θ is in radians. This complex number may then be expressed in exponential form if desired.

For single T scale

EXAMPLE: Find $\log_e (2.6 + j3.4)$. To convert $2.6 + 3.4$ to polar form, set right C– index over 3.4 on D. Move indicator to 2.6 on D. Read $\theta = 52.6°$ on T (read right to left) under hairline. Move slide to bring cos 52.6° on S (read right to left) under hairline, and read 4.28 on D under right index. Set indicator on 4.28 of the LL3+ scale. Read 1.454 on the DF/M scale. Then $\log_e (2.6 + j3.4) = 1.454 + j52.6°$, or $1.454 + j0.92$, when θ is in radians. This complex number may then be expressed in exponential form if desired.

READINGS BEYOND THE SCALES

Occasionally there is need to compute an expression which involves values not on the scales. To compute b^m for b less than 1.0023, note that by the binomial expansion $(1+xy)^{\frac{m}{y}} = 1 + mx + \dots$, and if xy is sufficiently small, these first two terms will give a good approximation.

From the theory of series it is known that $\log_e (1+x) = x - x^2/2 + x^3/3 +$ When x is small, say $x < 0.0025$, we may take $\log_e (1+x) = x$. Hence for such values of x the logarithm to base e may be set or read directly on DF/M. Moreover, for x small, $\log_{10}(1+x) = 0.4343\ x$, and hence if x is set on DF/M, $\log_{10} (1+x)$ will be found on D under the hairline. Thus $\log_e 1.000603 = 0.000603$, and $\log_{10} 1.000603 = 0.000262$.

Examples:

(a) Find $(1.0004)^{2.7}$. Since 1.0004 cannot be set on the scales, compute $1 + (2.7)(.0004) = 1.00108$, approximately.
The result can also be found easily by the following method. Set the left index of the C scale under 4 on DF/M, move the indicator over 27 on C, read 108 on DF/M. That is $\log_e (1.0004)^{2.7} = 2.7(0.0004) = 0.00108$. Hence $(1.004)^{2.7} = 1.00108$.

(b) Find $53^{0.00008}$. Although 53 can be set, the result cannot be read on the scales. Write the expression in the equivalent form $\left[53^{\frac{0.00008}{0.02}}\right]^{0.02} = (53^{0.004})^{0.02}$
The expression in brackets is found in the usual manner to be 1.016. Then $(1.016)^{0.02} = 1 + 0.02 \times 0.016 = 1.0003$, approximately.

(c) Find 30^8. The usual setting leads to a result beyond the LL4+ scale. Write the expression $5^8 \times 6^8$. Now $5^8 = 3.9 \times 10^5$, approximately, and $6^8 = 1.7 \times 10^6$ approximately. Hence $30^8 = 3.9 \times 10^5 \times 1.7 \times 10^6 = 3.9 \times 1.7 \times 10^{11} = 6.6 \times 10^{11}$. Moreover, note $30^8 = 30^4 \times 30^4 = 8.1 \times 10^5 \times 8.1 \times 10^5 = 66 \times 10^{10} = 6.6 \times 10^{11}$, approximately. Thus, by breaking up the expressions into factors, and computing each separately, the approximate results are obtainable. These results are also readily obtained by logarithms.

Also, it may be noted that if greater accuracy is desired in the logarithms of any numbers set on the LL4+ scale to the right of 10^3, these numbers may be set on the scale above (the LL3+ scale), and the sequence of digits in the mantissa read from the D scale. The characteristic is given by the primary scale division at the left of the setting on the LL4+ scale. Thus, to find the logarithm of 2,430,000, or 2.43×10^6, note that this number could be set on LL4 between 10^6 and 10^7. Set the hairline over 2.43 on LL3+ and read 385 on the D scale. Then, log 2,430,000 = 6.385, approximately.

The logarithms of numbers on the LL4– scale between 10^{-3} and 10^{-10} may also be obtained in this way. Thus, to find log .000000437, or log 4.37×10^{-7}, set 4.37 on the LL3+ scale and read 640 on the D scale. Then log .000000437 = 7.640 – 10 approximately.

ILLUSTRATIVE APPLIED PROBLEMS

1. **A volume of 1.2 cu. ft. of air at 60° F (or 520° absolute) and atmospheric pressure (14.7 lbs./sq. in.), is compressed adiabatically to a pressure of 70 lbs./sq. in. What is the final volume and final temperature?**

(a) Compute: $V = 1.2\left(\frac{14.7}{70}\right)^{\frac{1}{1.4}}$ Ans. 0.394 cu. ft.

Set 70 on C opposite 14.7 on D, read .21 on D under the C– index By means of the hairline, transfer .21 to the LL3– scale, and pull the right index under the hairline. Move hairline to 1.4 on CI, read .328 on LL3– under the hairline. Multiply 1.2 × .328 by the C and D scales, reading .394 on the D scale.

(b) Compute: $T=520\left(\frac{70}{14.7}\right)^{\frac{0.4}{1.4}}$ Ans. 812° Absolute or 352° F.

Divide 70 by 14.7, and set the result, 4.76, under the hairline on LL3+. Move right index of the C scale under the hairline, then move hairline over 0.4 on the C scale, then pull the slide so 1.4 of the C scale is under the hairline. Read 1.564 on LL3+. Multiply this by 520, obtaining 812, the final temperature in degrees absolute. Subtract 460° to obtain 352°F.

2. (a) Find the compound amount on an investment of $1200 at 3½% compounded annually for 20 years. The formula is $A = P(1+i)^n$, or, in this example, $A = 1200(1.035)^{20}$. Set left index of the C scale on 1.035 on LL2+. Move hairline over 20 on the C scale, read 1.99 on LL3+. Multiply this by 1200, obtaining $2390, approximately.

(b) In how many years does money double itself at 4.2% compounded annually? This problem requires finding n in the expression $(1.042)^n = 2$. Set the left index of the C-scale over 1.042 on LL2+, move hairline over 2 on LL3+, read 17 years, approximately, on the C scale under the hairline.

3. The formula $y = \frac{k}{1+be^{-at}}$ is the so-called "logistic of population."

For the United States, the time t is measured in years from 1780. From studies by the statistician Hotelling, $a = 0.0315$, $b = 64.5$, $k = 195.9$ (millions). Estimate the population for the year 1960 when the value of t will be 180.

Here $y= \frac{195.9}{1+64.5\times e^{-0.0315\times 180}}$

First compute $-.0315\times 180=-5.65$. Set the hairline over 5.65 on DF/M, read .0035 on LL4–. Multiply this by 64.5, obtaining .225, approximately. Add 1, and then divide 195.9 by 1.225, obtaining 160 (million) approximately, as the estimated population for 1960.

PART 5. HYPERBOLIC FUNCTIONS OF REAL VARIABLES

Hyperbolic functions are found useful in the application of mathematics to varied types of problems, and in particular, to problems in electrical en-

gineering. Computations involving these functions are readily performed on the Model 4 slide rule which has special scales for this purpose.

The most important hyperbolic functions may be defined as follows. Let x be any real number and e the base of Napierian logarithms. Then:

$$\frac{e^x - e^{-x}}{2} = \sinh x \text{ ("the hyperbolic sine of } x\text{")};$$

$$\frac{e^x + e^{-x}}{2} = \cosh x \text{ ("the hyperbolic cosine of } x\text{")};$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x \text{ ("the hyperbolic tangent of } x\text{")}.$$

EVALUATING THE FUNCTIONS

The Hyperbolic Sine

The scales marked Sh on the slide represent values of x ranging from $x = 0.10$ to $x = 3.0$, approximately The two scales may be viewed as one continuous scale which has been cut in half with the right hand portion placed below the left portion.

***Rule:* When the indicator is set over x on an Sh scale, the corresponding value of sinh x is on the C scale under the hairline, and conversely. If the C and D scales coincide, sinh x may also be read on the D scale. If x is found on the *upper* Sh scale, the decimal point is at the left of the number as read on the C scale. In other words, $0.1 \leq \sinh x \leq 1$. If x is found on the *lower* Sh scale, the decimal point is at the right of the first digit read on the C scale. In other words, $1.0 \leq \sinh x \leq 10.0$.**

EXAMPLES:

(a) Find sinh 0.116. Set hairline over 0.116 on the upper Sh scale. Read 0.1163 on the C scale (or D scale when the indices coincide).
Verify that: sinh 0.274 = 0.277; sinh 0.543 = 0.570; sinh 0.951 = 1.100; sinh 1.425 = 1.960; sinh 2.84 = 8.53

(b) Find x if sinh x = 0.425. Set hairline of indicator over 0.425 on the C scale, read 0.413 on the upper Sh scale.
Verify that if sinh x = 6.38, then x = 2.552.

The Hyperbolic Tangent.

The scale marked Th on the slide represents values of x ranging from $x = 1.0$ to $x = 3.0$.

***Rule:* When the indicator is set over x on the Th scale, the corresponding value of tanh x is on the C scale under the hairline. The decimal point is at the left of the number as read on the C scale. In other words, the approximate limits are $0.1 \leq \tanh x \leq 1.0$. For values of x greater than 3, tanh x = 1.000 to a close approximation; the error is less than one-half of 1% and decreases rapidly.**

EXAMPLES:

(a) Find tanh 0.176. Set the indicator over 0.176 on the Th scale, read 0 174 on the C scale.
Verify that: tanh 0 236 = 0.232; tanh 0.528 = 0.484; tanh 1.145= 0.816

(b) Find x if $\tanh x = 0.372$. Set indicator over 0.372 on the C scale. Read $x = 0.391$ on the Th scale.

The Hyperbolic Cosine.

No special scale for the hyperbolic cosine is needed. From the definitions, $\tanh x = (\sinh x)/(\cosh x)$, and hence $\cosh x = (\sinh x)/\tanh x$. This suggests the following rule:

***Rule:* With C and D indices coinciding, set indicator over x on the Sh scale. Move slide until x on the Th scale is under the hairline. Read $\cosh x$ on the D scale under the C index.**

It will be observed that the first step in this rule sets the value of $\sinh x$ on the D scale. The second step sets the value of $\tanh x$ on the C scale in position for the division. The result of the division is then read on the D scale. Viewed in another way, $\sinh x$ may be multiplied by $1/\tanh x$. This reciprocal is automatically set on the CI scale in the second step of the rule above. For all values of x on the Sh scale, $1 < \cosh x \leqq 10.00$.

EXAMPLES:

(a) Find cosh 0.240. With the C and D scales coinciding, set the hairline over 0.240 on the upper Sh scale. Move the slide until 0.240 on Th is under the hairline. Read $\cosh 0.240 = 1.029$ on the D scale at the C index.

(b) Find cosh 1.62. With C and D scales coinciding, set the hairline on 1.62 on the lower Sh scale. Move the slide until 1.62 on Th is under the hairline. Read $\cosh 1.62 = 2.63$ on the D scale at the C index.

It follows from the definitions that $\cosh^2 x - \sinh^2 x = 1$, and hence $\sinh x = \sqrt{\cosh^2 x - 1}$. If the value of $\cosh x$ is given, and x is to be found, this formula may be used to convert the problem to the corresponding case for the hyperbolic sine.

EXAMPLE:

Given $\cosh x = 1.31$, find x. Since $\sinh x = \sqrt{1.31^2 - 1} = \sqrt{1.716 - 1} = \sqrt{0.716} = 0.846$, when the hairline is set on 0.846 of the C scale, $x = .768$, may be read on the upper Sh scale.

COMPUTATIONS INVOLVING HYPERBOLIC FUNCTIONS

Computations involving hyperbolic functions are easily performed by usual methods (e.g., use of the C and D scales) by setting the values of the functions on the appropriate scales.

EXAMPLES:

(a) Find $y = 24.6 \sinh 0.35$. Set the left index of the C scale opposite 24.6 on the D scale. Move indicator to 0.35 on the upper Sh scale. Read $y = 8.79$ on the D scale under the hairline.

(b) Find $y = 86.4 \tanh 0.416$. Set the right index of C on 86.4 of D. Move indicator to 0.416 on Th. Read $y = 34.0$ on D under the hairline.

(c) Find $y = 77.3 \cosh 1.26$. In this case, it is best to set cosh 1.26 first. With C and D scales coinciding, set indicator on 1.26 of Sh. Move slide so that 1.26 of Th is under the hairline. Move indicator to 77.3 of the C scale. Read $y = 147.$ on D.

(d) Compute $17.9 \sinh 0.317 \times \sin 22°$. With C and D indices coinciding, set indicator on 0.317 of the upper Sh scale. Turn rule over and move slide so the right index of the C scale is under hairline. Move indicator to 22° on the S scale. Pull slide until 179 of the CI scale is under hairline. Read 2161 on D scale under the C index. The decimal point is found by noting that, approximately, $\sinh 0.317 = 0.3$, $\sin 22° = .4$, and hence $\sinh 0.317 \times \sin 22°$ is about 0.12 or ⅛. Then $17.9 \times$ ⅛ is about 2. Hence the result is 2.161.

Other Hyperbolic Functions

By definition, the following relations hold:

$1/\tanh x = \coth x$ ("the hyperbolic cotangent of x")
$1/\cosh x = \operatorname{sech} x$ ("the hyperbolic secant of x")
$1/\sinh x = \operatorname{csch} x$ ("the hyperbolic cosecant of x").

Since the values of these three additional functions are the reciprocals of functions discussed earlier, $\coth x$ and $\operatorname{csch} x$ may be read directly on the CI scale. After $\cosh x$ has been set on the D scale, $\operatorname{sech} x$ may be read on the DI scale on the front side of the rule, or if the indices coincide, on the CI scale of either side.

EXAMPLES:

Verify that $\coth 0.49 = 2.2$, $\operatorname{csch} 0.49 = 1.96$, $\operatorname{sech} 0.49 = 0.891$.

LARGE AND SMALL VALUES OF THE ARGUMENT

For values of x greater than 3, both $\sinh x$ and $\cosh x$ are approximately equal to $e^x/2$. Hence, if x is set on DF/M, then e^x may be read on LL4+ and divided by 2 mentally. As noted above, $\tanh x$ in this case is approximately 1.

For small values of the argument x, the hyperbolic sine is approximately equal to x. Consequently, in computations involving $\sinh x$ for $x < 0.10$, no special scales are needed. The value of x may be set directly on a C or D or other appropriate scale and the computation continued. The same is true of the hyperbolic tangent. Moreover, the hyperbolic cosine for $x < 0.10$ is approximately equal to 1.

In evaluating hyperbolic functions of complex arguments, values of the circular sine and tangent less than 0.57° are sometimes needed. Although these values can be found by use of the special graduations for this purpose, it is usually more convenient to read the ST scale as though the decimal point were at the left of the numbers printed, and to read the C (or D, CI, DI, etc.) scale with the decimal point one place to the left of where it would normally be. Thus, $\sin 0.2° = 0.00349$; $\tan 0.16° = 0.00279$, read on the C scale.

EXAMPLES:

(a) Find $\sqrt{\sinh 0.073}$. Set the indicator over 73 on the D scale of the front side of the rule. Read 0.27 on the upper square root scale. Then $\sqrt{\sinh 0.073} = 0.27$.

(b) Find log tanh 0.06. Set indicator over .06 on scale LL4−. Read −1.222 on the D scale.

PART 6. HYPERBOLIC FUNCTIONS OF COMPLEX ARGUMENTS

The definitions of the hyperbolic functions may easily be extended to include cases in which the independent variables, or arguments, are complex numbers. Let z represent any complex number $x + jy$, where x and y are real numbers. Then:

$$\frac{e^z - e^{-z}}{2} = \sinh z, \quad \frac{e^z + e^{-z}}{2} = \cosh z, \quad \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh z.$$

By use of the definitions and the formula $e^{jz} = \cos z + j \sin z$ the following relations may be verified:

(1) $(e^{jz} - e^{-jz})/2j = \sin z$; $(e^{jz} + e^{-jz})/2 = \cos z$
(2) $\sinh z = -\sinh(-z) = -j \sin jz$
(3) $\cosh z = \cosh(-z) = \cos jz$
(4) $\sinh jz = j \sin z$
(5) $\cosh jz = \cos z$
(6) $\cosh^2 z - \sinh^2 z = 1$
(7) $\sinh z = \sinh(x + jy) = \sinh x \cosh jy + \cosh x \sinh jy$
$\quad = \sinh x \cos y + j \cosh x \sin y$
(8) $\cosh z = \cosh(x + jy) = \cosh x \cosh jy + \sinh x \sinh jy$
$\quad = \cosh x \cos y + j \sinh x \sin y$
(9) $\tanh z = \tanh(x + jy) = \sinh(x + jy)/\cosh(x + jy)$
(10) $\sin z = \sin(x + jy) = \sin x \cosh y + j \cos x \sinh y$
$\cos z = \cos(x + jy) = \cos x \cosh y - j \sin x \sinh y$
(11) $\sinh(-z) = -\sinh z$
$\cosh(-z) = \cosh z$
$\tanh(-z) = -\tanh z$

For particular values of z each of these functions is, in general, a complex number which may be regarded as a vector expressible in either the component form or in exponential form, $\rho\,\underline{/\theta}$.

Sometimes the complex number z is given in exponential or polar form $\rho\underline{/\theta}$; for example, $\sinh \rho\underline{/\theta}$, or in particular, $\sinh 2.4\underline{/15^\circ}$. In this case z may be expressed in the form $x + jy$ by means of the relations $x = \rho \cos \theta$, $y = \rho \sin \theta$. Thus $\sinh 2.4\underline{/15^\circ} = \sinh(2.315 + j\,0.622.)$

CHANGING SINH Z FROM COMPONENT TO EXPONENTIAL FORM

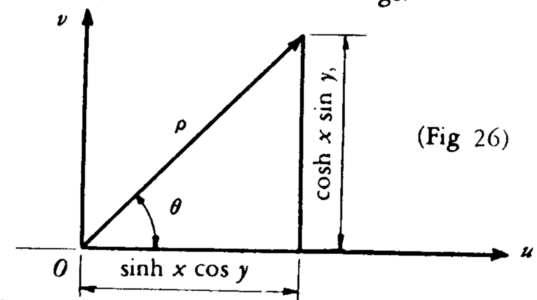
By formula (7) above, the complex number or vector

$$\sinh z = \sinh x \cos y + j \cosh x \sin y$$

is expressed in component form. If, for simplicity, u and v are defined by the formulas

$$u = \sinh x \cos y$$
$$v = \cosh x \sin y,$$
then $\sinh z = u + jv$.

A geometric representation of this complex number may be made by means of a u-axis of real numbers and a v-axis of pure imaginaries. The polar coordinates (ρ, θ) have their usual meanings.



(Fig 26)

Methods of changing to the polar form $\rho\underline{/\theta}$ by use of the slide rule will now be explained. First, it should be noted that the real number y may be expressed in either radians or angular degrees. Since the graduations on the S and T scales are in terms of degrees, this measure is more convenient. A value of y given in radian measure should therefore first be converted to angular degrées.

To recall the formula for sinh z readily, notice the following analogies: For real variables,

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
is similar in *form* to
$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$
For the complex variable,
$$\sinh(x + jy) = \sinh x \cos y + j \cosh x \sin y,$$
there are *formal* similarities, but the operator j serves to replace the functions of y by ordinary *circular* functions. Thus, *x is always associated with hyperbolic functions, while y is associated with circular functions.*

The following relations (See Fig. 26) are basic to the computations:

I (a) $$\tan \theta = \frac{v}{u} = \frac{\cosh x \sin y}{\sinh x \cos y} = \frac{\tan y}{\tanh x}$$

(b) $$\rho = \frac{\sinh x \cos y}{\cos \theta}$$

Observe that the ratio for tan θ involves a function of y divided by a function of x, and is thus analogous to $\tan \theta = y/x$ for circular functions.

***Finding ρ for sinh** $(x+jy)$*

Although it is usually better to find θ first, the explanations are long. The exposition is simplified by first considering a method of finding ρ assuming θ is known. If θ has been found first the value of ρ may be computed by the following rule, based on formula I (b).

Rule for ρ: **With C and D indices coinciding**
first, set the hairline over x on an Sh scale, and turn the rule over;
second, move the slide until θ on the S scale (read from *right to left* for the cosine) is under the hairline;
third, move the hairline to y on the S scale (read from *right to left*); read ρ on the D scale under the hairline.

Note that the first step sets sinh x on the C scale, the second step divides this by cos θ, and the third step multiplies by cos y. All the operations are actually done on the C and D scales, but only the final result needs to be read on D. The values of sinh x, cos y, and cos θ are automatically set.

EXAMPLES:

(a) Find ρ for sinh $(0.48 + j\,17°)$, given that $\theta = 34.4°$. With C and D indices together, set hairline over $x = 0.48$ on Sh. Move slide until 34.4 on S (reading from right to left) is under hairline. Move indicator to 17 on S (reading from right to left). Read $\rho = 0.578$ on D under hairline.

(b) Find ρ for sinh $(1.4 + j\,40°)$, given that $\theta = 43.4°$. With indices together, set indicator over 1.4 on Sh. Move slide until 43.4 on S (reading right to left) is under hairline. Move indicator to 40 on S (right to left). Read $\rho = 2.05$ on D under hairline.

(c) Find ρ for sinh$(0.73 + j2.2°)$, given $\theta = 3.53°$. With C and D indices together, set hairline over $x = 0.73$ on Sh. The settings for cos 3.53 and cos 2.2 are so near the right end of the slide that practically no change from the original setting is observable. In other words, for $y = 2.2°$, the value of $v = \cosh x \sin y$ is near zero. The value of cos y is near 1, and ρ is approximately equal to $u = \sinh 0.73$. Hence $\rho = 0.797$.

Finding θ

The ratio of the "pure imaginary" component to the real component determines the tangent of θ, and hence θ, as shown in formula I (a). The general rule is as follows.

Rule for θ: **To find θ for sinh $(x + jy)$ in the form $\rho\underline{/\theta}$:**
first, with C and D indices together, set hairline over y on a T scale;
second, move slide until x on Th is under hairline;
third, move indicator to C index;
fourth, move slide until C and D indices are together. Read θ on a T scale under the hairline.

The determination of the T scale on which θ is to be read depends upon the decimal point in the value of tan θ. This value may be noted on the D scale at the C index. However, to determine the decimal point, it is well to make a mental note of the approximate values of tan y and tanh x as they are set. By taking only the first digit, a mental computation easily gives the decimal point in the value of tan θ.

The following cases may arise.

Rule:

(i) If $0.01 < \tan\theta \leqq 0.1$, then $0.573° < \theta \leqq 5.71°$ on ST
(ii) If $0.1 < \tan\theta \leqq 1.0$, then $5.71° < \theta \leqq 45°$ on upper T
(iii) If $1.0 < \tan\theta \leqq 10.0$, then $45° < \theta \leqq 84.3°$ on lower T

The following cases may also occasionally occur:

(iv) If $10.0 < \tan\theta$, then $84.3° < \theta < 90°$. Since $\tan\varphi = \cot\theta$, where $\varphi = 90 - \theta$, the angle φ may be read on the ST scale and then $\theta = 90 - \varphi$.
(v) If $0 \leq \tan\theta \leq 0.01$, then $0 \leq \theta \leq 0.573°$. Read angle on ST, and divide by 10; that is, move decimal point one place to left of ST reading.

Step one sets the value of tan y on the D scale. Step two automatically sets the value of tanh x on the C scale for the division. The quotient is on the D scale at the C index. Since this is the value of tan θ, when the indices are brought together in step four the value of θ on a T scale is under the hairline. If desired, the following can be substituted for steps three and four above.

Third, read D scale at C index and move indicator over this value on C.
Fourth, read θ on a T scale under the hairline.

Although these last two rules may appear easier to use than the others, it should be remembered that to start finding ρ by the rule given earlier the indices must be together. Thus, the rule as originally given ends with the slide in position to begin finding ρ. Other methods are given later.

Since $\tanh x < 1$, it follows from the relation $\tan\theta = \tan y/\tanh x$ that $\theta > y$ for all values of $y < 90°$. Moreover, since $\tanh x \rightarrow 1$ as x becomes larger, the difference $\theta - y$ becomes smaller as x increases. These observations are sometimes useful as a rough check on θ. Thus, for sinh $(1 + j75°)$ the value of θ is 78.45° (Note 78.45>75); for sinh $(2 + j75°)$, $\theta = 75.5°$; for sinh $(1 + j81°)$, $\theta = 83.15°$, and for sinh $(2 + j81°)$, $\theta = 81.34°$.

Each of the following examples should be followed through several times to gain familiarity with the method and to observe how quickly the calculation can be completed when the details of the explanation are omitted.

EXAMPLES:

(a) Find θ for sinh $(0.48 + j\,17°)$. In this case, $\tan\theta = \tan 17°/\tanh 0.48$. Set the slide so that C and D scales coincide. Move indicator to 17 on T. Note on the D scale that tan 17 is about 0.3. Turn rule over and move slide until $x = 0.48$ on Th is under hairline. Note on the C scale that tanh 0.48 is about 0.4, and hence tan θ is roughly $0.3/0.4 = ¾ = .75$. Actually, it is 0.685, read on the D scale at the C index. This is case (ii). Move the indicator to the C index, bring the C and D indices together, and read $\theta = 34.4$ on the T scale under the hairline. The value of $\rho = 0.578$ was found in Example (a) page 72. Hence $\rho\underline{/\theta} = 0.578\,\underline{/34.4°}$.

(b) Find θ for sinh $(0.73 + j\,2.2°)$. Here $\tan\theta = \tan 2.2/\tanh 0.73$. With C and D scales coinciding, set hairline over 2.2° on ST. Observe that $\tan 2.2° = 0.04$, approximately. Turn rule over, and move slide so 0.73 on Th is under hairline. Note $\tanh 0.73 = 0.6+$ on the C scale. Hence $\tan\theta = 0.04/0.6$ roughly, or 0.0616, which can be read on the D scale at C index. This is case (i). Move indicator to C-index, and bring indices together. Read $\theta = 3.53°$ on ST. The value of ρ was found in Example (c) page 72. Hence $\rho\underline{/\theta} = 0.797\,\underline{/3.53°}$.

(c) Find $\rho\underline{/\theta}$ for sinh $(0.55 + j\,1.5°)$. Make C and D indices coincide, move hairline to 1.5° on ST. Note tan 1.5° is roughly 0.025 (on C or D). Move slide so 0.55 on Th is under hairline. Note tanh (0.55) is 0.5 on C. Hence $\tan\theta = 0.025/0.5$, or tan θ is about 0.05 (on D at C index). This is case (i). Move indicator to C index, bring indices together, and read $\theta = 3°$ on ST under hairline. In this case, ρ is approximately equal to sinh 0.55, or $\rho = 0.58$. Hence $\rho\underline{/\theta} = 0.58\,\underline{/3°}$.

(d) Find $\rho\underline{/\theta}$ for sinh $(0.19 + j\,4.2°)$. With C and D scales together, set hairline on 4.2° of ST. Move slide so that 0.19 on Th is under hairline. Observe that, roughly, $\tan\theta = 0.07/0.19$ or about 0.4. This is case (ii). Move indicator to C index, and bring indices of C and D to coincidence.

Read $\theta = 21.36°$ on T under hairline. Set hairline over 0.19 on Sh. Move slide until 21.36° on S (read right to left) is under hairline. Move indicator to 4.2° near right index of S. Read $\rho = 0.201$ on D under hairline. Hence sinh $(0.19 + j4.2°) = 0.204 \underline{/21.36°}$.

(e) Find $\rho \underline{/\theta}$ for sinh $(0.424 + j\ 38°)$. With C and D indices together set hairline over 38° on T. Note tan 38° is about 0.8. Move slide until 0.424 on Th is under hairline. Note on C that tanh 0.424 is about 0.4. Then tan $\theta = 1.95$ (on D at C index). This is case (iii). Move indicator to index, bring indices together, and read $\theta = 62.88°$ on lower T. To find ρ, with C and D indices coinciding, set indicator over 0.424 on Sh, move slide to bring cos 62.88° under hairline, move indicator to cos 38° on S. Read $\rho = 0.756$ on D. Hence $\rho\underline{/\theta} = 0.756 \underline{/62.88°}$.

(f) Find θ for sinh $(0.31 + j58°)$. With C and D scales together, set indicator to 58° on T. Move slide until 0.31 of Th. is under hairline. Note on D that tan $\theta = 5.34$. Move indicator to right C-index, then move slide so C and D indices coincide, and read $\theta = 79.37°$ on T. With C and D indices together, set hairline on 0.31 of Sh, pull slide until 79.37 on S (read right to left) is under hairline, move indicator to 58° on S (read right to left), read $\rho = 0.905$ on D. Hence, sinh $(0.31 + j58°) = 0.905 \underline{/79.37°}$

(g) Find θ for sinh $(0.31 + j75°)$. With C and D scales together, set indicator to 75° on T. Move slide so 0.31 of Th. is under hairline, then move hairline on left C-index.

Note that tan $\theta = 12.4$ on D. This is case (iv). Since cot θ is on the C scale above the D index, read $\varphi = 4.6°$ on ST. Hence $\theta = 90° - 4.6° = 85.4°$. To find ρ, with indices together set $x = 0.31$ on Sh, move slide until 4.6 on ST is under hairline, then move indicator to 75° on S (read right to left). Read $\rho = 1.014$ on D. Hence, $\rho\underline{/\theta} = 1.014 \underline{/85.4°}$.

(h) Find θ for sinh $(1.4 + j80°)$. With C and D scales together, set indicator to 80° on T. Move slide until 1.4 of Th is under hairline. Note on D that tan $\theta = 6.4$. Move indicator to right C index, then move slide so C and D indices coincide, and read $\theta = 81.1°$ on T. With indices together, set hairline on sinh 1.4. Move slide until cos 81.1° is under hairline. Move hairline over cos 80°. Read $\rho = 2.14$. Hence $\rho/\theta = 2.14 \underline{/81.1°}$.

ALTERNATIVE METHODS FOR SINH *Z*

The alternative methods of treating sinh z described below have certain advantages and also certain disadvantages. If the methods outlined above are used, the following will serve as checks.

(i) The formulas $u = \sinh x \cos y$; $v = \cosh x \sin y$ may be used to compute u and v. Then $\sinh z = u + jv$ may be converted to exponential form by ordinary vector methods. In this case, three major steps are required.

EXAMPLE:

For $\sinh(0.48 - j17°), u = 0.477, v = 0.327; u + v = 0.477 + j0.327$. Using the method described in example (a), page 79, $\theta = 34.4°$, $p = 0.578$.

(ii) The formula I (a), page 79, $\tan \theta = \tan y/\tanh x$, suggests that θ may be obtained from the complex number.

(12) $r\underline{/\theta} = \tanh x + j \tan y$, although in general r is not equal to ρ (See Fig. 27)

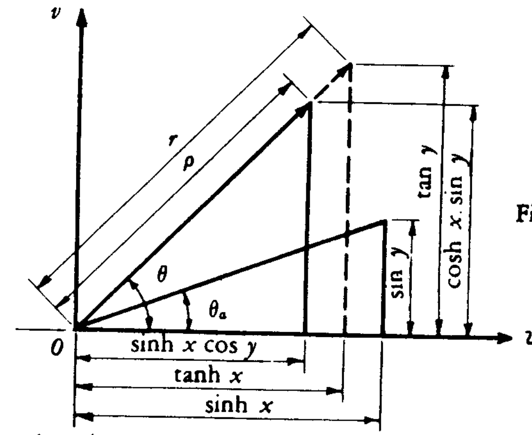


Fig. 27.

Moreover, the relation

(13) $\rho = \sqrt{\sinh^2 x + \sin^2 y}$

suggests that ρ may be obtained from the complex number

(14) $\rho\underline{/\theta_a} = \sinh x + j \sin y$, where $\tan \theta_a = \sin y/\sinh x$. It follows from I (a), page 71, that $\tan \theta = \dfrac{\cosh x}{\cos y} \tan \theta_a$.

Since, for all values of $x \neq 0$ and of $y \neq 0$, $(\cosh x) > 1$ and $(\cos y) < 1$, the ratio $(\tan \theta/\tan \theta_a) > 1$ and consequently $\theta > \theta_a$. From Fig. 27,

(15) $\cos \theta = \dfrac{\sinh x \cos y}{\rho} = \dfrac{\tanh x}{r}$,

and therefore $\rho = r \cosh x \cos y$. These results lead to the following methods.

The value of ρ may be obtained from formula (14) by ordinary vector methods and then θ found by use of (15).

EXAMPLE:

Let $\sinh (0.48 + j\ 17°) = \rho\underline{/\theta}$. Then

$$\rho\underline{/\theta_a} = \sinh 0.48 + j \sin 17°$$
$$= 0.4986 + j\ 0.292$$
$$= 0.578 \underline{/30.4°} \text{ (Note: } \theta_a = 30.4 < \theta).$$

This calculation is most readily carried out if the slide rule is equipped with *two* indicators. With C and D indices together, set one indicator on 0.48 on Sh, the other on 17° of S. Move C index to the larger reading on C, read $\theta_a = 30.4°$ on T under the hairline. Then move slide until 30.4° on S is under this hairline. Read ρ on D at C index. Using formula (15), since ρ is known, find $\theta = 34.4°$.

(iii) The angle θ may be found by ordinary vector methods from formula (12), and then p found by I (b), page 79, or (15) as explained earlier.

EXAMPLE:

For sinh $(0.48 + j\,17°)$, $r/\underline{\theta} = \tanh 0.48 + j \tan 17°$. With indices together, set right indicator on 0.48 of Th, and left on 17° of T. Move C index to larger reading, and read $\theta = 34.4$ under other hairline. Find ρ by regular methods.

(iv) The value of ρ may be obtained from (13) and then θ found by use of (12) or (15).

In using any method, it is wise to give some attention to the approximate values being set on the scales, and to the operations being performed as suggested by the formulas. In the long run, it is probably best to adopt one method and use it almost exclusively, rather than to invite confusion and error by attempting a variety of methods.

CHANGING COSH Z FROM COMPONENT TO EXPONENTIAL FORM

By formula (8) page 70, the complex number or vector

$$\cosh z = \cosh x \cos y + j \sinh x \sin y.$$

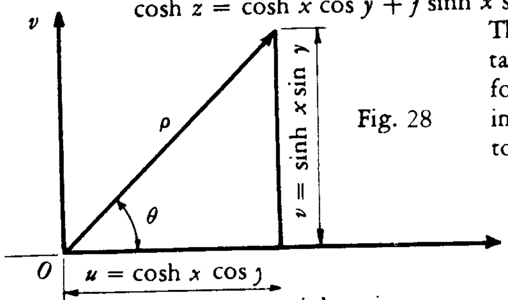


Fig. 28

The geometric representation is similar to that for sinh z. The following relations are basic to the computations:

$$\text{II (a)} \qquad \tan\theta = \frac{v}{u} = \frac{\sinh x \sin y}{\cosh x \cos y} = \tanh x \tan y$$

$$\text{(b)} \qquad \rho = \frac{\sinh x \sin y}{\sin \theta}$$

Note that I (b) for sinh z expresses ρ in terms of sinh x and the *cosines* of y and θ, while II (b) for cosh z expresses ρ in terms of sinh x and the *sines* of y and θ. Note also that tan θ above could be written $\tan y/\coth x$, but that this form is less convenient for computation. It should be noticed that, since $(\tanh x) < 1$, it follows that $\theta < y$ for the hyperbolic cosine.

Finding ρ for Cosh $(x + jy)$.

Assume θ has been found. The rule for ρ is the same as given earlier (page 71) for sinh z except the S scale is read from *left to right* for sines.

EXAMPLES:

(a) Find ρ for cosh $(0.31 + j\,58°)$ if $\theta = 25.7°$. With C and D indices together, set hairline on $x = 0.31$ of Sh. Move slide until 25.7° on S is under hairline, then move indicator to 58° on S. Read $\rho = 0.616$ on D under hairline.

(b) Find ρ for cosh $(1.4 + j\,80°)$ if $\theta = 78.7°$. Set sinh 1.4 on D at 1.904. Move slide so 78.7° on S is under hairline, then indicator to 80° on S. Read $\rho = 1.91$ on D. Notice that y and θ are so nearly equal that the ratio of their sines is near 1, and hence ρ is approximately equal to sinh 1.4.

Finding θ for Cosh $(x + jy)$

Formula (IIa) shows that tan θ can be found by a simple multiplication using the Th and T or ST scales with the D scale. The rules for determining on which scale (T or ST) the angle θ may be found are the same as those given earlier (page 72). However, since $\tanh x < 1$ and, for $y < 45°$, $\tan y < 1$, it follows that $\tan\theta < 1$, and cases (iii) and (iv) cannot arise for $y < 45°$.

Rule for θ

With C and D indices together, set indicator on x of Th. Move slide until an index of C (right or left as needed) is under hairline. Move indicator to y on T (or ST). Bring indices together, and read θ on T or ST.

EXAMPLES:

(a) Find ρ and θ for cosh $(0.23 + j\,16°)$. Here $\tan\theta = \tanh(0.23) \times (\tan 16°)$. With C and D indices together, set indicator over 0.23 on Th. Move slide so left C index is under hairline. Move indicator to 16° on T. Since $\tan\theta = (0.2)(0.3) = 0.06$, roughly, or, more accurately, 0.0648, this is case (i), and $\theta = 3.71° +$ is read on ST after the scales have been made to coincide. Now set indicator on 0.23 of Sh (about 0.23 on D), move slide until 3.71° on ST is under hairline (about 0.064 on C), change indices and move indicator to 16 on S (about 0.276 on C). Read $\rho = 0.99$ on D. The decimal point is determined by mentally calculating 0.23/0.06, which is about 4; ther 4×0.276 is about 1. Hence the result cannot be either 0.099 or 9.9, and mus be 0.99. Thus, cosh $= (0.23 + j\,16° = 0.99\,\underline{/3.71°}$.

(b) Find ρ and θ for cosh $(0.68 + j\,40°)$. With C and D indices together, set $x = 0.68$ on Th. Note (roughly) 0.6 on D. Turn rule over and move right C index under hairline. Move indicator to 40° on T. Note 0.8 + on C, and the product $(0.6)(0.8) = 0.496$ on D. This is Case (ii). Bring indices together and read $\theta = 26.4°$ on T. Set indicator over 0.68 on Sh. Move slide until 26.4 on S is under hairline. (Note quotient sinh 0.68/sin 26.4° is about 1.65). Multiply by sin 40°; reading 1.06 on D. Hence $\rho\,\underline{/\theta} = 1.06\,\underline{/26.4°}$.

(c) Find ρ and θ for cosh $(0.285 + j\,4.2°)$. With C and D indices together, set $x = 0.285$ on Th. Set right C index under hairline, then move indicator to 4.2° on ST. Note $.3 \times .07 = .021$, and hence this is case (i) With scales coinciding, read $\theta = 1.165°$ on ST. Calculate $\rho = 1.046$ Hence $\rho\,\underline{/\theta} = 1.046\,\underline{/1.165°}$.

(d) Find $\rho\,\underline{/\theta}$ for cosh $(0.2 + j\,1.3°)$. Note tan 0.2 is about 0.2; tan 1.3° is about 0.02. Hence tan θ is about 0.004, on scale D. This is case (v). When indices are together, read $\theta = 0.257°$ on ST with decimal point moved one place to the left. In this case, $\rho = 1.02$ approximately.

(e) Find $\rho/\underline{\theta}$ for cosh $(0.31 + j58°)$. With C and D indices together, set hairline on $x = 0.31$ on Th. Note $\tanh x = 0.3$. Move slide until left C-index is under hairline. Move hairline over 58° on T. Note tan 58 = 1.6 on C, and $\tan\theta = 0.48$ on D. This is case (ii). Bring scales to coincide and read $\theta = 25.7°$ on T. Since $\rho = 0.616$ for this problem was found in Example (a), page 77, $\rho/\underline{\theta} = 0.616\underline{/25.7°}$. Notice $\theta < y$ as a very rough check.

(f) Find $\rho/\underline{\theta}$ for cosh $(1.4 + j80°)$. Set tanh 1.4 on D. Notice the value is about 0.9. To multiply by tan 80°, move slide so right index of C is over tanh 1.4 on D, then move indicator over 80° on T. Notice tan 80° is near 6 on C. Hence tan $\theta = 5.02$ on D. This is case (iii). Move C and D index together and read $\theta = 78.7°$ on T. Note $\theta < y$. Since $\rho = 1.91$ was found in Example (b), page 77, $\rho/\underline{\theta} = 1.91 \,/\underline{78.7°}$.

(g) Find $\rho\,/\underline{\theta}$ for cosh $(2 + j87°)$. In this case, $y > 84.3°$. Hence tan 87° = cot (90-87) = cot 3°, or 19.1 read on CI opposite 3° on ST. Thus set tanh 2 on D, move slide so 3° on ST is over tanh 2 on D. Note at C-index that tan $\theta = 18.4$, and this is case (iv). Read $\varphi = 3.12°$ on ST at the D index, and thus $\theta = 90 - 3.12 = 86.88°$, which is, as it should be, less than $y = 87°$. To find ρ, set sinh 2 = 3.63 on D. Since θ and y are approximately equal, it follows that $\rho = 3.63$ approximately. Hence $\rho/\underline{\theta} = 3.63 \,/\underline{86.9°}$.

ALTERNATIVE METHODS FOR COSH Z

Other methods of computing ρ and θ for cosh $(x + jy) = u + jv$ are briefly outlined below:

(i) Compute numerical values of $u = \cosh x \cos y$ and $v = \sinh x \sin y$. Convert the complex number $u + jv$ to exponential form by ordinary vector methods.

(ii) The formula (IIa) written in the form

(16) $\tan\theta = \tanh x/\cot y = \tan y/\coth x$

suggests that θ may be obtained from either of the complex numbers

(17) $r_a\,/\underline{\theta} = \cot y + j\tanh x$, or $r_b\,/\underline{\theta} = \coth x + j\tan y$. Although in general $r_a \neq r_b \neq \rho$. Moreover, the relation

(18) $\rho = \sqrt{\sinh^2 x + \cos^2 y}$,

which may be verified by computing $\rho = \sqrt{u^2 + v^2}$ and using formula (6), suggests that ρ may be found from the complex number

(19) $\rho\,/\underline{\theta_a} = \sinh x + j\cos y$.

The θ_a in this formula satisfies the relation

$$\tan\theta = \frac{\sin y}{\cosh x}\cot\theta_a, \text{ and } \rho = r_a \cosh x \sin y.$$

By these and other possible formulas the problem may be reduced to one which may be solved by ordinary vector methods as described earlier.

EXAMPLE:

Let cosh $(0.31 + j58°) = \rho/\underline{\theta}$; then $u = \cosh 0.31 \cos 58 = 0.556$; $v = \sinh 0.31 \sin 58° = 0.267$. Hence $u + jv = 0.616 + j25.7°$, by methods described in example (e), page 78. If $r_a/\underline{\theta} = \cot 58° + j\tanh 0.31$, with the indices together set cot 58° on D by placing indicator over 32° on T. Move right C index to hairline then move indicator to tan h 0.31 = 0.3 on C. Move hairline over 0.3 on D and read 25.7° on T. Move slide so sin 25.7° on S is under hairline, read $r_a = 0.693$ on D at C index. If $\rho/\underline{\theta_a} = \sinh 0.31 + j\cos 58°$; set right index on C on cos 58° = 0.53. Move indicator to sin h 0.31 = 0.315 on C. Move hairline over 0.315 on D and read $\theta_a = 30.7°$ on T. Move slide so sin 30.7° on S is under hairline, read $\rho = 0.616$ on D at C index.

CHANGING TANH Z FROM COMPONENT TO EXPONENTIAL FORM

Although tanh $(x + jy)$ can be expressed in the form of $u + jv$, the formulas for u and v are not convenient for slide rule work. However, $\tanh z = \sinh z/\cosh z$ is expressible as a quotient. If the complex numbers sinh z and cosh z are expressed in the exponential form.

$$\sinh z = \rho_1/\underline{\theta_1}, \text{ and } \cosh z = \rho_2/\underline{\theta_2}, \text{ then}$$
$$\tanh z = (\rho_1/\rho_2)\,/\underline{\theta_1 - \theta_2}$$

Rule for Tanh z.

Express sinh z in the form $\rho_1/\underline{\theta_1}$.
Express cosh z in the form $\rho_2/\underline{\theta_2}$.
Then tanh $z = (\rho_1/\rho_2)\,/\underline{\theta_1 - \theta_2}$.

EXAMPLES:

(a) Find $\rho\,/\underline{\theta}$ for tanh $(0.31 + j58°)$.
From Example (f), page 74, sinh $(0.31 + j58°) = 0.905\,/\underline{79.37°}$
From Example (e), page 78, cosh $(0.31 + j58°) = 0.616\,/\underline{25.7°}$

$$\text{Hence } \tanh(0.31 + j58°) = \frac{0.905}{0.616}\,/\underline{79.37° - 25.7°}$$
$$= 1.47\,/\underline{53.67°}$$

(b) Find $\rho/\underline{\theta}$ for tanh $(1.4 + j80°)$.
From Example (h), page 74, sinh $(1.4 + j80°) = 2.14\,/\underline{81.1°}$
From Example (f), page 78, cosh $(1.4 + j80°) = 1.91\,/\underline{78.7°}$

$$\text{Hence } \tanh(1.4 + j80°) = \frac{2.14}{1.91}\,/\underline{81.1° - 78.7°}$$
$$= 1.12\,/\underline{2.4°}$$

ILLUSTRATIVE APPLIED PROBLEMS

1. Determine the "insertion loss" caused by inserting a line between a generator and a load using the formula

$$\frac{I_1}{I_2} = \cosh \gamma l + \frac{Z_g Z_r + Z_0^2}{(Z_g + Z_r) Z_0}\sinh \gamma l,$$

when $Z_g = 215$ ohms, $Z_r = 420$ ohms, $l = 150$ miles, $Z_0 = 720\,/\underline{-15°}$, $\alpha = 0.00720$ neper/mile, $\beta = 0.0280$ radian/mile, and $\gamma l = l(\alpha + j\beta)$.
First find

$\gamma l = 150\,(0.00720 + j\,0.0280) = 1.08 + j\,4.20 = 1.08 + j\,240.5°$

Since cosh γl is to be added to another complex expression, express it in component form.

$$\cosh(1.08 + j\,240.5°) = \cosh 1.08 \cos 240.5° + j\sinh 1.08 \sin 240.5°$$
$$= -\cosh 1.08 \cos 60.5° - j\sinh 1.08 \sin 60.5°$$

With indices together, set indicator on 1.08 of Sh. Move slide until 1.08 on Th is under hairline. Exchange indices, and move hairline over 60.5° on S (reading right to left). Find 0.809 on D. Then set sinh 1.08 = 1.30 + on the D scale and multiply by sin 60.5° using the S scale. Read 1.13 on D. Hence cosh $(1.08 + j\,240.5°) = -0.809 - j\,1.13$.

As an illustration of another method of finding cosh $(1.08 + j240.5^\circ)$, take $y = 240.5^\circ - 180^\circ = 60.5^\circ$, set indices together and indicator over 1.08 on Th. Move slide until right index is under hairline. Move indicator to 60.5° on T. Observe that $\tan\theta = 1.4$, bring indices together and read 54.5° on T under hairline. With indices together again set hairline on 1.08 of Sh. move slide until 54.5° on S is under hairline, then move hairline to 60.5° on S. Read $\rho = 1.393$ on D under hairline. Hence

$$\cosh(1.08 + j\,240.5^\circ) = 1.393\,\underline{/180^\circ + 54.5^\circ} = 1.393\,\underline{/234.5^\circ}$$
$$= 0.809 - j\,1.13 + .$$

Next compute the coefficient of $\sinh \gamma l$.

$$\frac{Z_g Z_r + Z_0^2}{Z_0(Z_g + Z_r)} = \frac{215 \times 420 + (720\,\underline{/-15^\circ})^2}{(215 + 420)\,720\,\underline{/-15^\circ}},$$
$$= \frac{90{,}300 + 518{,}000\,\underline{/-30^\circ}}{457{,}000\,\underline{/-15^\circ}},$$
$$= \frac{90{,}300 + 449{,}000 - j\,259{,}000}{457{,}000\,\underline{/-15^\circ}}$$
$$= \frac{539{,}000 - j\,259{,}000}{457{,}000\,\underline{/-15^\circ}}$$
$$= \frac{598{,}000\,\underline{/-25.65^\circ}}{457{,}000\,\underline{/-15^\circ}}$$
$$= 1.308\,\underline{/-10.65^\circ}$$

The coefficient just computed is to be multiplied by $\sinh\gamma l$, and hence it is best to express the latter in polar form. Except for algebraic signs, $\sinh(1.08 + j240.5^\circ) = \sinh(1.08) + j60.5^\circ)$. With indices coinciding, set hairline on 60.5 of T. Pull 1.08 of Th under hairline and move indicator to right index of C. Bring indices together. Read 65.8° on T under hairline. Set hairline over 1.08 of Sh. Move slide until 65.8° on S (reading right to left) is under hairline, then move indicator to 60.5° on S. Read $\rho = 1.565$ on D. Since the original angle of 240.5° was in the third quadrant, the angle found must be corrected by adding 180° Then $\sinh(1.08 + j240.5^\circ) = 1.565\,\underline{/245.8^\circ}$.

Multiplying the two factors gives

$$1.308\,\underline{/-10.65^\circ} \times 1.565\,\underline{/245.8^\circ} = 2.05\,\underline{/235.2^\circ}$$

In order to add this to the first term in the formula, the polar form should be replaced by the component form.

Compute $235.2^\circ - 180^\circ = 55.2^\circ$, and then $2.05 \cos 55.2 = 1.17$ and $2.05 \sin 55.2^\circ = 1.68$. Both results must be given a negative sign since 235.2° is in the third quadrant. Thus

$$\mathbf{I}_1/\mathbf{I}_2 = -0.809 - j\,1.13 + (-1.17 - j\,1.68)$$
$$= -1.979 - j\,2.81$$

This may be converted to polar form, noting that the angle is in the third quadrant. Thus

$$\mathbf{I}_1/\mathbf{I}_2 = 3.43\,\underline{/180^\circ + 54.8}$$
$$= 3.43\,\underline{/234.8^\circ},$$

approximately, and the absolute value is 3.43. Finally, compute $\log_e 3.43$ by setting 3.43 on scale LL3+and reading on DF the value 1.233, the "insertion loss" in nepers.

Although not a part of this problem, $\tanh(1.08 + j\,240.5^\circ)$ is easily obtainable from the above results and will be found to illustrate the method. Thus

$$\frac{\sinh(1.08 + j\,240.5^\circ)}{\cosh(1.08 + j\,240.5^\circ)} = \frac{1.565\,\underline{/245.8^\circ}}{1.393\,\underline{/234.5^\circ}} = 1.122\,\underline{/11.3^\circ}$$

It will prove to be interesting, and good practice in computation with the hyperbolic scales, to carry through the calculations for $l = 100$ miles and $l = 200$ miles, and to compare the results with those for $l = 150$ miles given above. Below are some results typical of those obtainable quickly without careful attention to accuracy.

If $l = 100$ mi., then $\gamma l = 0.72 + j\,2.8 \doteq 0.72 + j\,160^\circ$
$\cosh(0.72 + j\,160^\circ) = -1.193 + j\,0.268$, and
$\sinh(0.72 + j\,160^\circ) = 0.855\,\underline{/180^\circ - 30.5^\circ} = 0.855\,\underline{/149.5^\circ}$

The coefficient $1.308\,\underline{/-10.65^\circ}$ is unchanged, and
$1.308\,\underline{/-10.6^\circ} \times 0.855\,\underline{/149.5^\circ} = 1.118\,\underline{/138.9^\circ}$,
and this is equal to $-0.843 + j\,0.734$. Hence
$\mathbf{I}_1/\mathbf{I}_2 = 1.193 + j\,0.268 + (-0.843 + j\,0.734)$
$= -2.036 + j\,1.002 = 2.27\,\underline{/180^\circ - 26.2^\circ}$.
Then $\log_e 2.27 = 0.818$ nepers loss.
If $l = 200$ miles, $\gamma l = 1.44 + j\,320^\circ = 1.44 - j\,40^\circ$;
$\cosh(1.44 - j\,40^\circ) = 1.71 - j\,1.28$;
$\sinh(1.44 - j\,40^\circ) = 2.09\,\underline{/-43.2^\circ}$.
Then $1.308\,\underline{/-10.6^\circ} \times 2.09\,\underline{/-43.2^\circ} = 2.735\,\underline{/-53.8^\circ} = 1.61 - j2.21$.
Hence $\mathbf{I}_1/\mathbf{I}_2 = 3.32 - j3.49 = 4.82\,\underline{/-46.4^\circ}$, and $\log_e 4.82 = 1.57$ nepers loss.

2. Determine the voltage $\mathbf{V}_{ro} = \dfrac{\mathbf{V}_s}{\cosh S\sqrt{\mathbf{zy}}}$, at the end of an open circuit line in which the sending end voltage $V_s = 4$ volts, the length of the line $S = 50$ miles, and $\sqrt{\mathbf{zy}} = \alpha + j\beta$, where $\alpha = 0.0048$ neper/mile and $\beta = 0.0276$ radian/mile. Thus

$$\mathbf{V}_{ro} = \frac{4/0^\circ}{\cosh(0.24 + j\,1.38)}$$

First change 1.38 radians to degrees. Thus set π on CF opposite 180 on DF, move hairline to 1.38 on CF, and read 79° on DF under the hairline. To find cosh $(0.24 + j79°)$, first set indices together, and then set hairline over 0.24 on Th. Move slide to bring right index under hairline, then move indicator over 79° on T. Bring indices together and note $\tan\theta = 1.21 +$ on D, read $\theta = 50.5°$ on T.

With C and D indices together, set hairline on 0.24 of Sh. Move slide until 50.5° on S is under hairline, then set hairline on 79° of S. Read $\rho = 0.308$ on D. Then

$$V_{ro} = \frac{4\ \underline{/0°}}{0.308\ \underline{/50.5°}} = 13\ \underline{/-50.5°} \text{ volts, approximately.}$$

PART 7. INVERSE HYPERBOLIC FUNCTIONS OF COMPLEX ARGUMENTS

If ρ and θ are known, and $z = x + jy$ is to be determined so that $\sinh z = \rho\underline{/\theta}$, an inverse problem is involved. Similar problems arise involving $\cosh z$ and $\tanh z$.

Finding z for sinh $z = \rho\underline{/\theta}$.

To find $z = x + jy$ when $\sinh z = \rho\underline{/\theta}$, or $z = \operatorname{arc\,sinh} \rho\underline{/\theta}$, first write $\sinh z = \rho\cos\theta + j\rho\sin\theta = u + jv$ where $u = \rho\cos\theta$ and $v = \rho\sin\theta$ are known numbers. Since

$\sinh(x + jy) = \sinh x\cos y + j\cosh x\sin y$, equating the real and the imaginary components in the two expressions for sinh z, one has the equations

$$(20)\qquad \begin{aligned}\sinh x\cos y &= u\\ \cosh x\sin y &= v,\end{aligned}$$

from which x and y are to be found. Now by use of equation (6),

$v^2 = (1 + \sinh^2 x)\sin^2 y$, and from (20), $\sinh^2 x = u^2/\cos^2 y$.

On eliminating x and replacing $\cos^2 y$ by $1 - \sin^2 y$, the equation

$$\sin^4 y - (u^2 + v^2 + 1)\sin^2 y + v^2 = 0$$

is obtained. This is a quadratic in $\sin^2 y$, and hence

$$\sin^2 y = \left\{(u^2 + v^2 + 1) \pm \sqrt{(u^2 + v^2 + 1)^2 - 4v^2}\right\}/2,$$

in which the positive sign must be discarded. Then $\sqrt{2}\sin y = \sqrt{p - \sqrt{q}}$, where $p = u^2 + v^2 + 1$ and $q = p^2 - 4v^2$. In algebra it is shown that an irrational expression of this form can be written in the simpler form $\sqrt{U_1} \pm \sqrt{V_1}$ provided $p^2 - q$ is a perfect square. Since in this case $p^2 - q = 4v^2$, the condition is fulfilled. Thus U_1 and V_1 are to be found such that

$$\sqrt{p \pm \sqrt{q}} = \sqrt{U_1} \pm \sqrt{V_1}$$

Squaring, and equating first the rational and then irrational parts, on solving the two equations thus obtained one has

$$U_1 = \{u^2 + (v + 1)^2\}/2;$$
$$V_1 = \{u^2 + (v - 1)^2\}/2.$$

Hence it follows that

$$\text{III (a)}\qquad \sin y = \frac{U_s - V_s}{2}, \text{ where}$$

$$U_s = \sqrt{u^2 + (v + 1)^2} \text{ and } V_s = \sqrt{u^2 + (v - 1)^2},$$

from which y can be found in terms of u and v. The negative sign is chosen because otherwise the value of $\sin y$ would exceed 1. From equations (20)

$$\text{III (b)}\qquad \sinh x = u/\cos y,$$

from which x may be found.

It is convenient to regard U_s as the resultant of a vector whose components are u and $v + 1$; and similarly, to regard V_s as the resultant of a vector whose components are u and $v - 1$.

Then for convenience writing

$$U_s\underline{/\theta_a} = u + j(v + 1) \text{ and } V_s\underline{/\theta_b} = u + j(v - 1),$$

one may compute U_s and V_s by the methods described on page 83 .

In order that the meaning of the process may be clarified the example which follows is the inverse of one solved earlier.

Example.

Find $z = x + jy$ so that $\sinh z = 0.201\ \underline{/21.36°}$
First find $u = 0.201\cos 21.36 = 0.187$, and
$v = 0.201\sin 21.36 = 0.0734$.
Then $U_s\underline{/\theta_a} = 0.187 + j\,1.074 = 1.09\ \underline{/81°+}$ or $U_s = 1.09$;
$V_s\underline{/\theta_b} = 0.187 + j\,0.9266 = 0.945\ \underline{/78.6°-}$ or $V_s = 0.945$

Therefore $\sin y = \dfrac{1.09 - 0.945}{2} = .0725$, and hence $y = 4.15°$, which is found by setting .0725 on D and reading y on ST. To find x, find $\sinh x = 0.187/\cos 4.15° = 0.187/0.997 = 0.188$ or approximately 0.19. Hence $x = 0.19$ and $y = 4.2$ are the approximate values of x and y Compare with Example (d), page 73.

Finding z for cosh $z = \rho\underline{/\theta}$.

To find $z = x + jy = \operatorname{arccosh}\rho\underline{/\theta}$, where ρ and θ are known, first find $u = \rho\cos\theta$ and $v = \rho\sin\theta$. Since

$$\cosh z = \cosh x\cos y + j\sinh x\sin y,$$

the equations

$$\begin{aligned}\cosh x\cos y &= u\\ \sinh x\sin y &= v\end{aligned}$$

are to be solved for x and y. In this case it turns out to be more convenient to solve for $\cos y$ by the method described for $\sin y$ above. The results are as follows:

$$\text{IV (a)} \qquad \cos y = \frac{U_c - V_c}{2}, \text{ where}$$

$$U_c = \sqrt{(u+1)^2 + v^2} \text{ and } V_c = \sqrt{(u-1)^2 + v^2}.$$

$$\text{(b)} \qquad \sinh x = v/\sin y$$

Thus $\cosh z$ may be treated by methods essentially similar to those for $\sinh z$ outlined above.

Example:

If $\cosh z = 1.06 \underline{/26.4°}$, find $z = x + jy$.
First compute $u = 1.06 \cos 26.4° = 0.949$ and
$v = 1.06 \sin 26.4° = 0.472$. Then

$$U_c = \sqrt{1.949^2 + 0.472^2} \text{ or } 2.01$$
$$V_c = \sqrt{0.051^2 + 0.472^2} \text{ or } 0.475$$

Using formula IV (a)

$$\cos y = \frac{2.01 - 0.475}{2} = 0.767 \text{ and hence } y = 40°.$$

Using formula IV (b)

$$\sinh x = 0.472/\sin 40° = 0.735.$$

With the indicator on this value on C, turn the rule over, bring indices together, and read $x = 0.68$ on Sh. Hence $z = 0.68 + j\,40°$. Compare with Example (b), page 77.

Finding z for tanh z $= \rho\underline{/\theta}$.

If ρ and θ are known and $z = x + jy = \text{arc} \tanh \rho\underline{/\theta}$ is to be found, first find $u = \rho \cos \theta$ and $v = \rho \sin \theta$. Then $\tanh (x + jy) = u + jv$. It can readily be shown by use of the relation

$$(21) \quad \tanh (z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$$

that $\tanh (x - jy) = u - jv$, and hence

$$(22) \qquad \begin{aligned} x + jy &= \text{arctanh } (u + jv) \\ x - jy &= \text{arctanh } (u - jv). \end{aligned}$$

Adding,

$$2x = \text{arctanh } (u + jv) + \text{arctanh } (u - jv);$$

and by using equation (21)

$$\tanh 2x = \frac{u + jv + u - jv}{1 + (u + jv)(u - jv)}, \text{ or}$$

$$\text{V (a)} \qquad \tanh 2x = \frac{2u}{1 + u^2 + v^2} = \frac{2u}{1 + \rho^2}$$

Subtracting the equations (22),

$$2\,jy = \text{arctanh } (u + jv) - \text{arctanh } (u + jv), \text{ and hence}$$

$$\tanh 2jy = \frac{u + jv - (u - jv)}{1 - (u + jv)(u - jv)} = \frac{2jv}{1 - (u^2 + v^2)}$$

Finally, $\tanh 2jy = j \tan 2y$, and consequently

$$\text{V(b)} \qquad \tan 2y = \frac{2v}{1 - \rho^2}$$

Therefore x and y may be found from formulas V(a) and V(b), respectively. Expressed explicitly, they are

$$x = \tfrac{1}{2} \text{ arctanh } 2u/(1 + \rho^2);\ y = \tfrac{1}{2} \text{ arctan } 2v/(1 - \rho^2).$$

The values of z for $\coth z = \rho\underline{/\theta}$, $\text{sech } z = \rho\underline{/\theta}$, and $\text{csch } z = \rho\underline{/\theta}$ may be found, if desired, by using the reciprocal relationships; that is, by finding z for $\tanh z = (1/\rho)\underline{/-\theta}$, $\cosh z = (1/\rho)\underline{/-\theta}$, and $\sinh z = (1/\rho)\underline{/-\theta}$.

ILLUSTRATIVE APPLIED PROBLEM

Suppose the propagation constant $\gamma = \alpha + j\beta$ of a line is to be found by the formula

$$\tanh \gamma l = \sqrt{\frac{\mathbf{Z}_{ss}}{\mathbf{Z}_{so}}},$$

where $\mathbf{Z}_{ss} = 3520 \underline{/-86.3°}$ ohms, and $\mathbf{Z}_{so} = 1430 \underline{\ 72.7°}$ ohms, and $l = 30$ miles. Then

$$\tanh 30\gamma = \sqrt{\frac{3520 \underline{/-86.3}}{1430 \underline{/72.7°}}}$$
$$= 1.568 \underline{/-79.5°} = \rho\underline{/\theta}.$$

To calculate this, set 143 of C opposite 352 on D and read 1.568 on the upper $\sqrt{}$ scale at the C index. Then calculate $(-86.3 - 72.7)/2 = -79.5$. Move the left C index to 1.568 on D, move indicator to 79.5 on S (read right to left), and find u = 0.286. Move right C- index to 1.568 on D, indicator to 79.5° on S, and read 1.54, on D. Thus, since θ is in the fourth quadrant, $v = -1.54$ and $\tanh 30\gamma = 0.286 - j1.542 = u + jv$

$$\text{Now by (Va), } \tanh 2x = \frac{2\,(0.286)}{1 + 1.568^2} = \frac{0.572}{1 + 2.46} = \frac{0.572}{3.46} = 0.1653$$

Set hairline on 0.1653on C,read $2x = 0.1668$ on Th, from which $x = 0.0834$. From formula (Vb)

$$\tan 2y = \frac{2\,(-1.54)}{1 - 2.46} = \frac{-3.08}{-1.46} = 2.11$$

Set 2.11 on C and read 64.6° on lower T. Thus $2y = 64.6°$ or $64.6 + 180° = 244.6°$; then $y = 32.3°$ or $122.3°$. Suppose that other known conditions about the line suffice to determine that $y = 122.3°$ is the proper value. Then $30\gamma = 0.0834 + j122.3°$, or, since $122.3° = 2.135$ radians, $30\gamma = 0.0834 + j\,2.135$, whence $\gamma = 0.00278 + j\,0.0712$. To change this to polar form, set right C index on 712 of D, move hairline to 279 of D, and note that $0.0712/0.00278 = 25$ approximately. Hence read $\varphi = 2.24°$ on ST, and find $\theta = 90 - \varphi = 87.76°$. The value of this ρ is nearly 0.0712, or about 0.0713. Hence

$$\gamma = 0.0713 \underline{/87.76°}.$$

PART 8. CIRCULAR FUNCTIONS OF COMPLEX ARGUMENTS

By means of the formulas on page 70 the following relations may be proved:

$\sin(x + jy) = \sin x \cosh y + j \cos x \sinh y,$

$\cos(x + jy) = \cos x \cosh y - j \sin x \sinh y.$

The analogies between these results and those for $\sin(x + y)$ and $\cos(x + y)$ in trigonometry should be carefully noted. Observe also that here the x is associated with a *circular* function and the y is associated with a *hyperbolic* function.

These formulas express the sine and cosine of a complex argument as a complex number in terms of its components.

If $\sin(x + jy)$ is to be expressed as a complex number in polar form, reasoning similar to that on page 71 leads to the analogous results:

VI (a) $$\tan\theta = \frac{\tanh y}{\tan x},$$

(b) $$\rho = \frac{\sin x \cosh y}{\cos\theta}.$$

Moreover, $\rho = \sqrt{\sin^2 x + \sinh^2 y}$, and this formula may be used as a check or as the basis of an alternative method of computing ρ from the expression $\rho \underline{/\theta_a} = \sin x + j \sinh y$.

Similarly, if $\cos(x + jy)$ is to be expressed as a complex number in polar form, first find ρ and θ for the conjugate variable $x - jy$ in $\cos(x - jy)$ by the following formulas:

VII (a) $$\tan\theta = \tan x \tanh y,$$

(b) $$\rho = \frac{\sin x \sinh y}{\sin\theta}$$

Then $\cos(x + jy) = \rho \underline{/-\theta}$.

Finally, $\tan(x + jy)$ may be readily found by finding the sine and cosine in polar form and taking the quotient.

The close analogy between these formulas for circular functions and the corresponding ones for hyperbolic functions enables the computer to use similar methods of calculation. In the case of *hyperbolic* functions, the x of $x + jy$ is associated with a hyperbolic function. Symbolically, x is preceded by an h, as in $\sinh x$. In the case of the *circular* functions, the y of $x + jy$ is associated with a hyperbolic function, or symbolically, is preceded by an h, as in $\sinh y$. An adjustment of algebraic sign is called for in the case of the cosine.

EXAMPLES:

(a) Find $\sin(30° + j\,0.48)$. With C and D indices together, set indicator on 0.48 of Th. Move slide until 30° on T is under hairline. Note $\tanh 0.48/\tan 30 = 0.774$, approximately. Move indicator to C-index, and then bring indices together, reading $\theta = 37.7°$ on T. Or, move indicator to 0.774 on C and read θ on T.

Set indicator on 0.48 of Sh; move slide until 0.48 on Th is under hairline; move indicator to sin 30° on S; move slide until $\theta = 37.7°$ on S (reading right to left) is under indicator. Read $\rho = 0.707$ on D at right C-index. Then $\sin(30° + j\,0.48) = 0.707 \underline{/37.7°}$.

(b) Find $\cos(30° + j\,0.48)$. With C and D indices together, set indicator on 0.48 of Th; move right index of slide under hairline, then move hairline to 30° on T. Bring indices together, and read $\theta = 14.45°$ on T.

Set indicator on 0.48 of Sh; move slide until 14.45° on S is under hairline; move indicator to 30° on S; read $\rho = 1$ on D. Then $\cos(30° + j\,0.48) = 1 \underline{/-14.45°}$.

(c) Find $\tan(30° + j\,0.48)$. From examples (a) and (b) above:

$$\tan(30° + j\,0.48) = \frac{\sin(30° + j\,0.48)}{\cos(30° + j\,0.48)} = \frac{0.707 \underline{/37.7°}}{1 \underline{/-14.45°}} = 0.707 \underline{/52.15°}$$

If $z = x + jy$ is to be found so that $\sin z = u + jv$ or $\cos z = u + jv$, methods similar to those outlined on page 83 yield the following formulas from which x and y may be found.

VIII (a) $$\sin x = \frac{U_c - V_c}{2}, \text{ where}$$

$$U_c = \sqrt{(u+1)^2 + v^2} \text{ and } V_c = \sqrt{(u-1)^2 + v^2}$$

(b) $$\sinh y = v/\cos x$$

IX (a) $$\cos x = \frac{U_c - V_c}{2}, \text{ where } U_c \text{ and } V_c \text{ are defined as above.}$$

(b) $$\sinh y = v/\sin x$$

X (a) $$\tan 2x = \frac{2u}{1 - \rho^2}$$

(b) $$\tanh 2y = \frac{2v}{1 + \rho^2}$$

EXAMPLES:

(a) Find $z = x + jy$ if $\sin z = 0.707 \underline{/37.7°}$

Here $u = 0.558$; $v = 0.431$

$$\sin x = \left\{\sqrt{1.558^2 + 0.431^2} - \sqrt{0.442^2 + 0.431^2}\right\}/2$$

$$= (1.617 - 0.617)/2 = 0.500$$

Then $x = 30°$, and $\sinh y = 0.431/\cos 30°$, $y = 0.48$

Thus $x + jy = 30° + j\,0.48$

(b) Find $z = x + jy$ if $\cos(x + jy) = 1 \,/\!-14.45°$
Here $u = 0.967$; $v = -0.2495$

$$\cos x = \left\{\sqrt{(1.967)^2 + (-0.2495)^2} - \sqrt{0.033^2 + (-0.2495)^2}\right\}/2$$

$$= (1.984 - 0.251)/2 = 0.866 = \cos 30°$$

$$\sinh y = \frac{-(-0.2495)}{\sin 30°} = 0.499$$

Then $x = 30°$, $y = 0.480$, and $z = 30° + j\,0.48$

(c) Find $x + jy$ if $\tan(x + jy) = 0.707 \,\underline{/52.15°}$
First find $u = 0.707 \cos 52.15° = 0.433$, and
$v = 0.707 \sin 52.15° = 0.557$.
Then $\tan(x + jy) = 0.433 + j\,0.557$, and $\rho^2 = 0.5$

$$\tan 2x = \frac{2\,(0.433)}{1 - 0.5} = 1.732$$

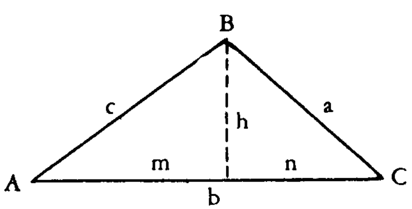
Hence $2x = 60°$ and $x = 30°$. Continuing,

$$\tanh 2y = \frac{2\,(0.557)}{1 + 0.5} = 0.743, \text{ which yields } 2y = 0.96 \text{ on Th,}$$

and hence $y = 0.48$. Finally, $x + jy = 30° + j\,0.48$ or $0.524 + j\,0.48$, if 30° is changed to radians.

PART 9—ONE-STEP SOLUTION OF "TWO SIDES AND INCLUDED ANGLE TRIANGLE" PROBLEM THROUGH USE OF DOUBLE T SCALE

The *"two sides and the included angle triangle"* problem can be solved by a single setting of the Model 2 or 4 rule. Using the following diagram, note that



h = c sin A = m tan A = n tan C = a sin C, m + n = b, and b is greater than c. This formula can be written in the form:

$$\frac{\sin A}{\frac{1}{c}} = \frac{\tan A}{\frac{1}{m}} = \frac{\tan C}{\frac{1}{n}} = \frac{\sin C}{\frac{1}{a}} = \frac{h}{1}$$

SOLUTION: Put *A* on S over *c* on DI; move hairline over *A* on T and read *m* on DI; subtract *m* from *b* giving *n*; over *n* on DI read *C* on T; under *C* on S read *a* on DI; over 1 on D read *b* on C.

It will frequently be necessary to interchange end numbers of the T scale since it is a double length scale. Since such an operation is not considered an operation on the rule, only a single setting has been made.

EXAMPLE:

Given c = 428, b = 537, A = 32.6°. Find C and a.

$$\frac{\sin 32.6°}{\frac{1}{428}} = \frac{\tan 32.6°}{\frac{1}{m}} = \frac{\tan C}{\frac{1}{n}} = \frac{\sin C}{\frac{1}{a}} = \frac{h}{1}$$

1. Put 32.6 on S over 428 on DI.
2. Under 32.6 on T read *m* (= 361) on DI. Then n = 537 − 361 = 176.
3. Interchange end numbers of the T scale.
4. Over 176 on DI read *C* (= 52.7°) on T. Depend on a diagram to know which part of the T scale to select.
5. Again interchange end numbers of the T scale.
6. Under 52.7 on S read *a* (= 290) on DI.
7. If *b* is desired, read *b* (= 231) on C over index of D.

HOW TO ADJUST YOUR SLIDE RULE

Each rule is accurately adjusted before it leaves the factory. However, handling during shipment, dropping the rule, or a series of jars may loosen the adjusting screws and throw the scales out of alignment. Follow these simple directions for slide rule adjustment.

ALIGNMENT OF RULE BODY

1. Position your slide rule so that the adjusting screws in the two end-plates are up and away from you.
2. Loosen the two end-plate screws to achieve slight flexibility in the rule.
3. Position the slider (or center part of the rule) so that its left index is aligned with the index on the fixed stator (at the **bottom** of the rule).
4. Keeping the slider aligned, position the movable stator (at the **top** of the rule) so that its index is aligned with the slider.
5. With thumb and forefinger, apply slight pressure to the left side of the rule, and tighten the screw. (Leave a small gap between the slider and stators to achieve smooth rule movement; approximately .003".)
6. Apply slight pressure to the right side of the rule and tighten that adjusting screw—again leaving a small gap.
7. Confirm alignment on the reverse side of the rule.

ALIGNMENT OF WINDOW ASSEMBLY

1. Loosen all screws on **both** sides of the window assembly to make the assembly flexible.
2. Working on one side of the rule, locate the **unsprung** cursor bar; e.g., the bar that **does not have** a tension spring. (Cursor bars are the opaque teflon components that ride on the edge of the rule.)
3. Using thumb and forefinger, apply upward pressure to the bottom of the unsprung bar—so that it rests firmly against the rule's edge.
4. Also position the window so that the hairline is perpendicular to the index lines on the left side of the rule. Tighten the screw(s) in the **unsprung** bar. **Leave the other screw(s) loose.**
5. Turn the rule over and repeat the preceding steps:
   a) Apply pressure against the cursor bar which has no spring.
   b) Make certain the hairline is perpendicular to the left index.
   c) Tighten the screw(s) in the unsprung bar.
6. Continuing on the same side of the rule, tighten the screw(s) in the cursor bar, which is spring-loaded.
7. Reverse the rule and tighten the remaining screw(s) in the spring-loaded bar.
8. Move the window to the opposite end of the rule to confirm alignment with the right-hand index.

REPLACEABLE ADJUSTING SCREWS All Pickett All-Metal rules are equipped with Telescopic Adjusting Screws. In adjusting your rule, if you should strip the threads on one of the Adjusting Screws, simply "push out" the female portion of the screw and replace with a new screw obtainable from your dealer, or from the factory. We do not recommend replacing only the male or female portion of the screw.

HOW TO KEEP YOUR SLIDE RULE IN CONDITION

OPERATION Always hold your rule between thumb and forefinger at the ends of the rule. This will insure free, smooth movement of the slider. Holding your rule at the center tends to bind the slider and hinder its free movement.

CLEANING Wash surface of the rule with a non-abrasive soap and water when cleaning the scales.

LUBRICATION The metal edges of your slide rule will require lubrication from time to time. To lubricate, put a little white petroleum jelly (Vaseline) on the edges and move the slider back and forth several times. Wipe off any excess lubricant. Do not use ordinary oil as it may eventually discolor rule surface.

LEATHER CASE CARE Your Leather Slide Rule Case is made of the finest top-grain genuine California Saddle Leather. This leather is slow-tanned using natural tanbark from the rare Lithocarpus Oak which grows only in California. It polishes more and more with use and age. TO CLEAN YOUR CASE and keep the leather pliable and in perfect condition, rub in a good harness soap.

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