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SUPPLEMENTARY MANUAL

LOG LOG DUPLEX VECTOR SLIDE RULE

REG. U. S. PAT. OFF.

No. N4083

NOTE: This booklet explains only the application of the plane and hyperbolic trigonometric scales to vector calculations. It is a supplement to the instructions covering the Log Log Duplex Decitrig Slide Rule No. N4081. *Be sure that you also receive the manual for No. N4081, which explains the general use of this slide rule.*

KEUFFEL & ESSER CO.

LOG LOG DUPLEX VECTOR

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Slide Rules *Measuring Tapes*

THE K & E LOG LOG DUPLEX VECTOR SLIDE RULE

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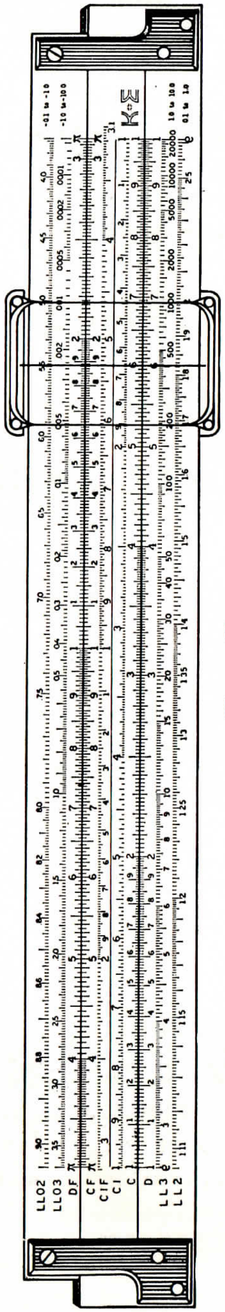

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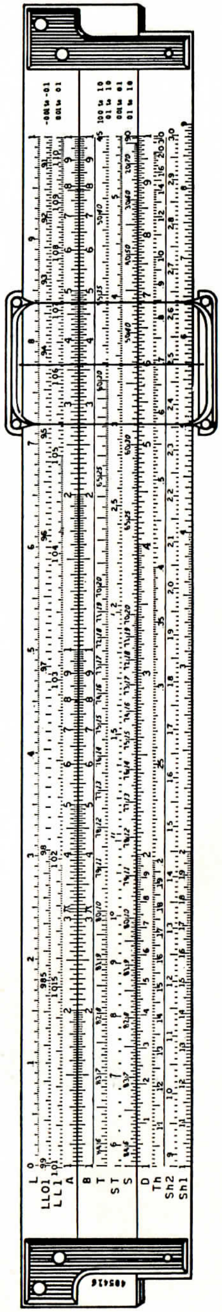

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PREFACE

The K & E Log Log Duplex Vector Slide Rule has all of the scales of the Log Log Duplex Decitrig Slide Rule with the exception of scales DI and K which are displaced to make room for the three scales of hyperbolic functions. Cubes and cube roots may be readily found by use of the log log scales; and it may be noted in the explanations given in the manual of the Log Log Duplex Decitrig Slide Rule, which accompanies this booklet, that the DI scale is not essential to the efficient employment of this rule.

It is suggested to the users of this slide rule that they should become thoroughly familiar with its application to calculating processes through thorough study of the instruction manual on the Log Log Duplex Decitrig Slide Rule. The latter which is also supplied with the Log Log Duplex Vector Slide Rule, includes a very thorough discussion of the solution of the right triangle on which vector calculations are based.

This manual on the Log Log Duplex Vector Slide Rule deals specifically with the general application of the trigonometric and hyperbolic scales to vector calculations, with examples illustrating their specific application to problems encountered in electrical engineering.

C O N T E N T S

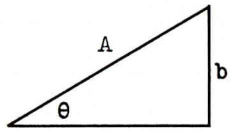
THE K & E LOG LOG DUPLEX VECTOR SLIDE RULE

I

PLANE VECTOR CALCULATIONS

(See also Pages 62 to 65 in the Manual of the Log Log Duplex Decitrig Slide Rule.)

Plane vectors of the form $Ae^{j\theta} = A\underline{/\theta}$, when subject to processes of addition or subtraction, must be converted into their horizontal and vertical components, jointly expressed in complex notation of the form a + jb. The quantity "a" is the horizontal component, "b" is the vertical component and the symbol $j = \sqrt{-1}$, called "operator", indicates that the component "b" is 90° in counter clockwise direction from the horizontal component "a" as shown in the figure.



The exponential form of vectors lends itself more conveniently to the processes of multiplication and division, involution and evolution. Hence when such calculations are to be performed on complex numbers of the form a + jb, it is necessary to convert them first into the exponential form.

The following rules of conversion are based upon the trigonometric relationship which exists between the components of the vector forming the sides of a right triangle, and the hypothenuse or absolute value of the vector.

(A) CONVERSION OF a + jb INTO $A\underline{/\theta}$ (See Fig. above).

Rule: Set an index of the slide to the larger side "a" (or "b") on scale D. Opposite the shorter side "b" (or "a") on scale D read angle "θ" on scale T. Move slide so as to bring angle "θ" on scale S opposite the shorter side, and at the slide index read "A" on scale D.

If side "a" is the larger side, angle "θ" is smaller than 45°; if side "a" is the smaller side, angle "θ" is greater than 45°. This should be taken into consideration when reading angle "θ" on scale T. In the first case the black numbering of scale T is used, whereas in the second case the red numbering of scale T is used.

Example 1:

Find the vector value of 4 + j3.

The vertical component is smaller than the horizontal component, therefore angle θ < 45°.
To 4 on scale D set right slide index.
At 3 on scale D read θ = 36.87° on scale T.
Move 36.87° on scale S to 3 on scale D.
At slide index read A = 5 on scale D.

Hence $4 + j3 = 5\underline{/36.87°}$

Example 2:

Find the vector value of 3 + j4.

Here the vertical component is greater than the horizontal component, therefore angle θ > 45°.
To 4 on scale D set right slide index.
At 3 on scale D read θ = 53.13° on scale T.
Move 53.13° on scale S (red) to 3 on scale D.
At slide index read A = 5 on scale D.

Hence $3 + j4 = 5\underline{/53.13°}$

Example 3:

Find the vector value for 9.68 ± j6.9.

θ < 45°.
To 9.68 on scale D set right slide index.
At 6.9 on scale D read θ = 35.5° on scale T.
Move 35.5° on scale S to 6.9 on scale D.
At slide index read A = 11.9 on scale D.

Hence $9.68 \pm j6.9 = 11.9\underline{/\pm 35.5°}$

Example 4:

1.91 + j0.05.

Since, $\frac{0.05}{1.91} < 0.1$, the angle "θ" can not be read on scale T, but will have to be read on scale ST.
To 1.91 on scale D set left slide index.
At 0.05 on scale D read θ = 1.5° on scale ST.
With the vector angle so small, the difference between the numerical value of the vector, "A", and the horizontal component will be so small that

$$1.91 + j0.05 = 1.91\underline{/1.5°}$$

for all practical purposes.

A more accurate value of "A" is

$$A = a + \frac{0.5\ b^2}{a} \text{ when } b < a, \text{ and}$$

$$A = b + \frac{0.5\ a^2}{b} \text{ when } b > a.$$

Thus for the preceding example we have

$$A = 1.91 + \frac{0.5\ (0.05)^2}{1.91} = 1.910654.$$

Example 5:

0.065 + j1.49.

Since, $\frac{1.49}{0.065} > 10$, the angle "θ" can not be read on scale T (red), and its value will have to be found on scale ST as follows:
To 1.49 on scale D set left slide index.
At 0.065 on scale D read 2.5° on scale ST.
90° - 2.5° = θ = 87.5°.

Since, 1.49 is comparatively large with respect to 0.065, it follows that for all practical purposes

$$0.065 + j1.49 = 1.49\underline{/87.5°}.$$

More accurately we get

$$A = 1.49 + \frac{0.5(0.065)^2}{1.49} = 1.491418.$$

(B) CONVERSION OF A/θ INTO a + jb.

Given $Ae^{j\theta} = A\underline{/\theta}$, find a + jb.
The complex expression for $A\underline{/\theta}$ is (A cos θ + jA sin θ).
Since scale D gives sin θ for values of "θ" on scale S (black). and cos θ for values of "θ" on scale S (red), the above notation can be evaluated by one setting of the scales S and D.

Rule: To "A" on scale D set an index of the slide. At "θ" on scale S (black) read the vertical, or imaginary component, and at "θ" on scale S (red) read the horizontal or real component.

Example 6:

Find the complex number equivalent to $5\underline{/36.87°}$.

To 5 on scale D set right slide index.
At 36.87° on scale S (black) read b = 3 on scale D.
At 36.87° on scale S (red) read a = 4 on scale D.

Hence $5\underline{/36.87°} = 4 + j3.$

Example 7:

Find the complex number equivalent to $11.9\underline{/-54.5°}$.

To 11.9 on scale D set left slide index.
At 54.5° on scale S (black) read b = 9.69 on scale D.
At 54.5° on scale S (red) read a = 6.91 on scale D.

Hence $11.9\underline{/-54.5°} = 6.91 - j9.69.$

Example 8:

Find the complex number equivalent to $1.83\underline{/2.5°}$.

To 1.83 on scale D set left slide index.
At 2.5° on scale ST read b = 0.0798 on scale D.
For all practical purposes

$$1.83\underline{/2.5°} = 1.83 + j0.0798.$$

As under the previous article we can find a more accurate value for "a" by the following formula:

$$a = A - \frac{0.5\ b^2}{A}$$

$$= 1.83 - \frac{0.5 \times (0.0798)^2}{1.83}$$

$$= 1.83 - 0.00174 = 1.82826$$

ILLUSTRATIVE APPLICATIONS

1. A circuit consists of a resistance of 7.46 ohms, in series with an inductive reactance of 3.4 ohms. What is the vector impedance of the circuit?

$$Z = 7.46 + j3.4 = 8.2\underline{/24.5°} \text{ vector ohms.}$$

2. Calculate the vector impedance of a circuit which consists of a resistance of 0.795 ohms in series with a capacitive reactance of 1.25 ohms.

$$Z = 0.795 - j1.25 = 1.48\underline{/-57.54°} \text{ vector ohms.}$$

3. A circuit consists of a resistance in parallel with a capacitance. The measured current in the resistance is 10.5 amperes, and that in the capacitance is 5.7 amperes. What is the total current in the circuit? What is its time phase with respect to the difference of potential impressed upon the circuit? What is the power factor?

$$I = 10.5 + j5.7 = 11.95\underline{/28.5°} \text{ vector amperes.}$$

The time phase is 28.5° leading.
The power factor is cos 28.5° = 0.878.

4. A circuit consists of a conductance of 0.068 mhos in parallel with a practically pure inductive susceptance of 0.0872 mhos. What is the vector admittance of the circuit?

$$Y = 0.068 - j0.087 = 0.1105\underline{/-52^\circ} \text{ vector mhos.}$$

5. The attenuation constant and the wave length constant of No.10 A.W.G. dry core cable at 800 cycles frequency are $a = 0.0366$ and $\beta = 0.0559$ hyperbolic and circular radians, respectively. What is the propagation constant of the line?

$$p = a + jb = 0.0366 + j0.0559.$$

$$= 0.067\underline{/56.8^\circ}.$$

6. The vector impedance of a circuit is $7.9\underline{/25.5^\circ}$ ohms. Calculate its resistive and reactive components.

$$7.9\underline{/25.5^\circ} = 7.13 + j3.4.$$

Hence $r = 7.13$ ohms and $x = 3.4$ ohms.

7. The admittance of a parallel circuit is $0.0955\underline{/-31.5^\circ}$ vector ohms. Calculate the conductance and susceptance of the circuit.

The conductance is: $g = 0.0955 \cos(-31.5^\circ) = 0.0814$ mhos,

and the susceptance is: $b = 0.0955 \sin(-31.5^\circ) = -0.0499$ mhos.

The complex expression for the admittance is : $g - jb = 0.0814 - j0.0499$

8. The vector propagation constant of a certain transmission line is

$$p = 0.065\underline{/56.5^\circ}.$$

Calculate the attenuation and phase constant.

The attenuation constant is:

$$a = 0.065 \cos 56.5^\circ = 0.03585 \text{ hyperbolic radians.}$$

The phase constant is:

$$\beta = 0.065 \sin 56.5^\circ = 0.0542 \text{ circular radians.}$$

The wave length constant is identical to the phase constant, but is usually expressed in degrees. Its value for the given line is:

$$0.0542 \times 57.3 = 3.11^\circ.$$
(One radian = 57.3°).

II

HYPERBOLIC FUNCTIONS

(A) THE HYPERBOLIC SINE AND TANGENT SCALES

1. Since for all practical purposes $\sinh x = x$, and $\tanh x = x$, when $x < 0.1$, scale D gives the sinh x and tanh x directly for any value of $x < 0.1$.

2. Scale D in conjunction with scale Sh1 gives the hyperbolic sines of hyperbolic angles x for any value between $0.1 < x < 0.882$. Thus for $x = 0.515$ set indicator on 0.515 scale Sh1 and read $0.538 = \sinh 0.515$ on scale D.

Similarly:
$$\sinh 0.645 = 0.69$$
$$\sinh 0.1945 = 0.1956$$
$$\sinh 0.273 = 0.276$$

Conversely, scale Sh1 gives the hyperbolic angle x corresponding to sinh x between values of 0.1 and 1.0. Thus for $\sinh x = 0.758$ on D, read $x = 0.7$ on Sh1 scale. For $\sinh x = 0.249$ on D, read $x = 0.2466$ on Sh1.

3. Scale Sh2 in conjunction with scale D gives the hyperbolic angle "x" for values of sinh x between values of 1 and 10 as read on scale D. Thus for $1.592 = \sinh x$ as read on scale D, we get $x = 1.245$ on scale Sh2. Conversely, scale D gives the corresponding values of sinh x for values of "x" between 0.882 and 3.0 on scale Sh2. Thus for $x = 1.245$ on scale Sh2, read $\sinh x = 1.592$ on scale D.

Similarly:
$$\sinh 1.465 = 2.05$$
$$\sinh 1.875 = 3.18$$
$$\sinh 2.24 = 4.64$$
$$\sinh 2.95 = 9.53$$

For: $\sinh x = 5.1$ on D read $x = 2.332$ on Sh2.
$\sinh x = 4.35$ on D read $x = 2.176$ on Sh2.
$\sinh x = 1.84$ on D read $x = 1.37$ on Sh2.

4. Scale Th gives the hyperbolic angle "x" for values of tanh x between 0.1 and 1.0, and conversely scale D gives the values of tanh x for values of "x" between 0.1 and 3.0. Thus for $x = 0.175$ on scale Th, we read 0.1733 on scale D. For values of "x" larger than 3.0, the hyperbolic tangent is substantially equal to 1.0.

Similarly:
$$\tanh 0.224 = 0.2202$$
$$\tanh 0.435 = 0.409$$
$$\tanh 0.94 = 0.735$$
$$\tanh 1.45 = 0.895$$
$$\tanh 2. = 0.964$$
$$\tanh 3. = 0.995$$

For: tanh x = 0.795 on D, read x = 1.085 on Th

tanh x = 0.52 on D, read x = 0.576 on Th

tanh x = 0.137 on D, read x = 0.138 on Th

(B) EVALUATION OF THE HYPERBOLIC COSINE

The hyperbolic cosine of any real number x, may be obtained by any of the following formulas:

$$\cosh x = \frac{\sinh x}{\tanh x} \qquad (1)$$

$$= \sqrt{1 + \sinh^2 x} \qquad (2)$$

or

$$= \frac{\sinh x}{\sin\,[\tan^{-1}\,(\sinh x)]} \qquad (3)$$

Formula (1) is best suited for slide rule calculation, and the value of cosh x, can be obtained at a single setting of the slide.

Rule: Set an index of the slide to "x" on scale Th. Move indicator to "x" on scale Shl or Sh2 and read cosh x on scale C, on the reverse side of the rule.

Example:

Find cosh 0.662

Set left index of slide to 0.662 on scale Th. Move indicator to 0.662 on scale Shl and read 1.227 = cosh x, on scale C.

Example:

Find cosh 1.57.

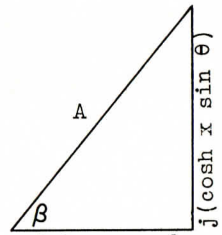
Set right index of slide to 1.57 on scale Th. Move indicator to 1.57 on scale Sh2 and read 2.508 = cosh x, on scale C.

III

HYPERBOLIC FUNCTIONS OF COMPLEX NUMBERS

(A) VECTOR EQUIVALENTS OF sinh (x + jθ).

The hyperbolic sine of a complex number is a complex number.



Thus $\sinh(x + j\theta) = (\sinh x \cos\theta) + j(\cosh x \sin\theta)$. (1)

The vector value of this expression is, therefore

$$\sinh(x + j\theta) = A\underline{/\beta}. \qquad (2)$$

where $A = \dfrac{\sinh x \cos\theta}{\cos\beta}$ (3)

or $A = \sqrt{\sinh^2 x + \sin^2\theta}$ (4)

and $\beta = \tan^{-1}\left(\dfrac{\cosh x \sin\theta}{\sinh x \cos\theta}\right) = \tan^{-1}\left(\dfrac{\tan\theta}{\tanh x}\right)$ (5)

$= \tan^{-1}\left(\dfrac{1}{\tanh x/\tan\theta}\right)$ (6)

$= \cot^{-1}\left(\dfrac{\tanh x}{\tan\theta}\right)$ (7)

$= \cot^{-1}(\tanh x \cot\theta)$ (8)

The numerical value of "A" may be determined through the relations given either by equation (3) or (4). Any of the last four equations may be used to determine the angle "β" of the function either directly, or through slide rule operation.

Since the value of the angle "θ" which enters into all the above expressions, is frequently stated in terms of radians when used in conjunction with hyperbolic functions of complex numbers, it is essential that this angle be first converted into degrees. The conversions may be performed easily and quickly by setting 180 on CF to π on DF. (See Pages 59 and 60 in the Manual of the Log Log Duplex Decitrig Slide Rule).

1. Slide Rule Method for Calculating the Numerical Value of sinh $(x + j\theta)$.

Equation (3) indicates that the numerical value of sinh $(x + j\theta)$ can be obtained easily, provided that the angle "β" of the function is determined first. The operation consists of a simple multiplication and division with the use of the Sh, S (red), and D scales as follows:

To "x" on scale Sh1 or Sh2 as required, set "β" on scale S (red).
Opposite "θ" on scale S (red), read "A" on scale D.

If in doing this, the angle "θ" is beyond the index of the rule body, exchange slide indexes and read "A" on D, opposite "θ" on S (red).

Equation (4) indicates that the numerical value of "A" may also be thought of as the hypothenuse of a right triangle whose sides are sinh x and sin θ.

For slide rule operation it may be written

$$A = \sinh x + j \sin \theta \qquad (4a)$$

The numerical value of the function may thus be calculated in a manner similar to that of a plane vector as explained in section I. This method does not demand that the angle "β" of sinh $(x + j\theta)$ be calculated first. It has the further advantage that it can be remembered easily. We must bear in mind, however, that the auxiliary angle that enters into the calculation of "A" by this method, is not the angle of the vector "A".

Whichever method is used to determine the value of "A", it is evident from equation (4) that its value is always larger than the largest of the two terms under the radical. Thus for sinh $(0.256 + j10.5°)$, for instance, the value of "A" is larger than sinh 0.256 = 0.259, while for sinh $(0.256 + j30°)$, the value of "A" is larger than sin 30° or 0.5.

It appears from what has just been said that a casual comparison of the magnitudes of the two terms under the radical of equation (4) is useful in locating the decimal in the reading of the value of "A" on scale D.

2. Slide Rule Method for Calculating the Angle "β" of sinh $(x + j\theta)$.

Equation (5) indicates that we may think of tan θ and tanh x, in their relationship to the angle "β", as the sides of a right triangle with "β" as the angle between one of the sides and the hypothenuse of the triangle:

$$\underline{/\beta} = \tanh x + j \tan \theta \qquad (5a)$$

It should be fully realized that in equation (5a) we are concerned only with the angle "β", and not in the numerical value of the apparent vector relation between the tanh x and tan θ.

From what has just been said, it follows that the slide rule method for obtaining the value of the angle "β" of sinh $(x + j\theta)$ is fundamentally identical to that of obtaining the angle of a plane vector as explained in section I provided that, instead of "x" and "θ", tanh x and tan θ were known. Thus we may obtain the angle "β" for all values of "x" and "θ" by first obtaining the values of "a" and "b" corresponding to tanh x and tan θ respectively, on scale D. Set slide index on the larger component on scale D, and read the angle "β" on scale T over the smaller component. This is the most direct method for obtaining the angle "β".

Example 1:

Evaluate "β" of sinh $(0.256 + j10.5°)$.

Solution: tanh 0.256 = 0.25
tan 10.5° = 0.185

To 0.25 on scale D set right slide index
Move indicator to 0.185 on scale D and
read "β" = 36.5° on scale T (black).

Very definite and simple rules which eliminate the necessity of ascertaining first the values of tanh x and tan θ, and based upon the above method are formulated below.

It should be observed from equation (3) in its relation to the tangent scales ST, T (black) and T (red), that the angle "β" of the sinh function may be smaller than 5.75°, in which case it must be read on scale ST; it may also be between 5.75° and 45° on scale T (black); or it may be larger than 45° on scale T (red). Furthermore, the value of angle "θ" may also be on any of these three scales. It is therefore important to determine a method for ascertaining the scale on which the angle "β" should be read.

The following cases may occur:

CASE 1. $\theta < 5.75°$; $\beta \lesseqgtr 5.75°$.

Since $\tanh x < 1$ and $0.01 < \tan\theta < 0.1$, it follows that $0.01 < \tan\beta < 1$. The angle "β" may be, therefore either smaller or greater than 5.75°, and thus lie either on scale ST or on scale T (black). It is just equal to 5.75° when $\tan\theta = 0.1 \tanh x$. Hence with slide and body matched, $\tan\theta < 0.1 \tanh x$ when "θ" on scale ST, is to the left of "x" on scale Th, the angle $\beta < 5.75°$ should be read on the scale ST. On the other hand, when "θ" is to the right of "x" on scale Th, $\tan\theta > 0.1 \tanh x$, and the angle $\beta > 5.75°$ should be read on scale T (black).

Another way of stating this is as follows:

Set indicator over "x" on scale Th, and move "θ" on scale ST under indicator. If slide protrudes to the right, $\beta < 5.75°$, and should be read on scale ST. If the slide protrudes to the left, $\beta > 5.75°$, and should be read on scale T (black).

To determine the value of the angle "β".
To "x" on scale Th, set "θ" on scale ST. At body index read "β" on ST, if slide protrudes to the right; or on scale T (black), if slide protrudes to the left.

Example 2:

Evaluate $\sinh(0.216 + j3.5°)$.

To find the angle "β":

Note that θ = 3.5 is to the right of 0.216 on scale Th.

To x = 0.216 on scale Th set θ = 3.5° on scale ST.
At body index read β = 16° on T (black).

To find the numerical value of the function:

To x = 0.216 on scale Shl set β = 16° on scale S (red)
Opposite 3.5° on scale S (red), read A = 0.226 on scale D.

Hence $\sinh(0.216 + j3.5°) = 0.226\underline{/16°}$.

Example 3:

Evaluate $\sinh(0.6 + j2.5°)$.

To find the angle "β":

Note that 2.5 on scale ST is to the left of 0.6 scale Th.

To x = 0.6 on scale Th set θ = 2.5 on scale ST.
At body index read β = 4.66° on scale ST.

To find the numerical value of the function:

To x = 0.6 on scale Shl set β = 4.66° on scale S (red).
Opposite θ = 2.5° on scale S (red)
read A = 0.638 on scale D.

Hence $\sinh(0.6 + j2.5°) = 0.638\underline{/4.66°}$.

The above example shows that for all practical purposes, when both "θ" and "β" are less than about 5°, the numerical value of the sinh function of a complex number is substantially equal to sinh "x".

CASE 2. $5.75° < \theta < 45°$; $\beta \lesseqgtr 45°$.

Since $\tanh x < 1$ and $0.1 < \tan\theta < 1$, it follows that $0.1 < \tan\beta < 10$. The angle "β" may be, therefore, either smaller or greater than 45°, and thus lie either on scale T (black) or on scale T (red). The angle "β" is just equal to 45° when $\tan\theta = \tanh x$. Hence with slide and body matched, $\tan\theta < \tanh x$, when "θ" on scale T (black), is on the left of "x", on scale Th. Consequently, the angle $\beta < 45°$ should be read on scale T (black). When angle "θ" is to the right of "x" on scale Th, $\tan\theta > \tanh x$, and the angle $\beta > 45°$ should be read on scale T (red).

It follows, therefore, that the rule given under Case 1 holds also for Case 2. If with angle "θ" on scale T (black) over "x" on scale Th, the slide protrudes to the right, the angle $\beta < 45°$ should be read on scale T (black); and if the slide protrudes to the left, the angle $\beta > 45°$ should be read on scale T (red).

To determine the value of "β", we must first ascertain whether it is smaller or larger than 45°.

(2a) $5.75° < \theta < 45°$ and $\beta < 45°$.

To "x" on scale Th, set "θ" on scale T (black).
Opposite body index read "β" on scale T (black).

Example 4:

Evaluate $\sinh(0.256 + j10.5°)$.

Solution:
To x = 0.256 on scale Th, set θ = 10.5° on scale T (black)
At body index read β = 36.5° on T (black).

The numerical value of the function is obtained by the rule set forth in section (A) 1. Thus

To $x = 0.256$ on scale Shl set $\beta = 36.5°$ on scale S (red).
Opposite $\theta = 10.5°$ on scale S (red)
read $A = 0.316$ on scale D.

Hence sinh $(0.256 + j10.5°) = 0.316\underline{/36.5°}$.

Example 5:

Evaluate sinh $(1.28 + j30.5°)$.

Solution:
To $x = 1.28$ on scale Th, set $\theta = 30.5°$ on scale T (black).
Opposite body index read $\beta = 34.5°$ on scale T (black).

To find the numerical value of the function:

To $x = 1.28$ on scale Sh2 set $\beta = 34.5°$ on scale S (red)
Opposite $\theta = 30.5°$ on scale S (red)
read $A = 1.735$ on scale D.

Hence sinh $(1.28 + j30.5°) = 1.735\underline{/34.5°}$.

Example 6:

Evaluate sinh $(0.505 + j20°)$.

Solution:
To $x = 0.505$ on scale Th, set $\theta = 20°$ on scale T (black).
Opposite body index read $\beta = 38°$ on scale T (black).

To find the numerical value of the function:

To $x = 0.505$ on scale Shl set $\beta = 38°$ on scale S (red).
Opposite $\theta = 20°$ on scale S (red).
read $A = 0.628$ on scale D.

Hence sinh $(0.505 + j20°) = 0.628\underline{/38°}$.

(2b) $5.75° < \theta < 45°$ and $\beta > 45°$.

To obtain the angle "β" for this case: with slide and body matched, set indicator over "θ" on scale T (black), move left slide index under indicator and opposite "x" on scale Th read "β" on T (red).

Another method which may be used to advantage is as follows:

To "x" on scale Th, set "θ" on scale T (black). At slide index read cot β on scale D. Opposite this reading on scale C, read angle β on T (red).

Example 7:

Evaluate sinh $(0.43 + j30.5°)$.

With slide matched to body, set indicator over $\theta = 30.5°$ on scale T (black).
Move right slide index under indicator,
Opposite $x = 0.43$ on scale Th,
read $\beta = 55.47°$ on scale T (red).

Using the second rule:

To $x = 0.43$ on scale Th set $\theta = 30.5°$ on scale T (black).
At slide index read cot $\beta = 0.688$.
Opposite 0.688 on scale C
read $\beta = 55.47°$ on scale T (red).

To find the numerical value of the function:

To $x = 0.43$ on scale Shl set $\beta = 55.47°$ on scale S (red).
Opposite $\theta = 30.5°$ on scale S (red)
read $A = 0.674$ on scale D.

Hence sinh $(0.43 + j30.5°) = 0.674\underline{/55.47°}$.

Example 8:

Evaluate sinh $(0.17 + j35°)$.

With slide index matched to body index, set indicator over $\theta = 35°$ on scale T (black).
Move right slide index under indicator,
Opposite $x = 0.17$ on scale Th
read $\beta = 76.47°$ on scale T (red).

Using the second rule:

To $x = 0.17$ on scale Th, set $\theta = 35°$ on scale T (black).
At slide index read cot $\beta = 0.2405$.
Opposite 0.2405 on scale C
read $\beta = 76.47°$ on scale T (red).

To find the numerical value of the function:

To $x = 0.17$ on scale Shl, set $\beta = 76.47°$ on scale S (red
Opposite $\theta = 35°$ on scale S (red)
read $A = 0.598$ on scale D.

Hence sinh $(0.17 + j35°) = 0.598\underline{/76.47°}$.

CASE 3. $\theta > 45°$; $\beta > 45°$.

Referring to equation (3) it is seen that since $\tanh x < 1$ and $\tan \theta > 1$, the angle of the sinh function will always be larger than 45° and should be read on scale T (red).

The method of obtaining "β" is:

To right body index set "θ" on scale T (red).
At "x" on scale Th read "β" on scale T (red).

Another method which may be used is as follows:

Set slide index to "x" on scale Th.
Opposite "θ" on T (red), read cot β on scale D.
Opposite this value on scale C, read "β" on T (red).

Example 9:

Evaluate $\sinh (0.68 + j62°)$.

To right body index set "θ" = 62° on scale T (red).
Opposite x = 0.68 on scale Th
read β = 72.54° on scale T (red).

To find the numerical value of the function:

To x = 0.68 on scale Shl set β = 72.54° on scale S (red).

We find that θ = 62° on scale S (red) is beyond the right index, therefore exchange indexes and

At θ = 62° on S (red), read A = 1.147 on scale D.

Hence $\sinh (0.68 + j62°) = 1.147\underline{/72.54°}$.

Example 10:

Evaluate $\sinh (0.243 + j53.3°)$.

To right body index set θ = 53.3° on scale T (red).
Opposite x = 0.243 on scale Th
read β = 79.93° on scale T (red).

To find the numerical value of the function:

To x = 0.243 on scale Shl set β = 79.93° on scale S (red).
At θ = 53.3° on scale S (red), read a = 0.839 on scale D.

Check the following examples:-

$\text{Sinh} (0.63 + j30.5°) = 0.842\underline{/46.57°}$.

$\text{Sinh} (0.17 + j55°) = 0.837\underline{/83.28°}$.

$\text{Sinh} (1.7 + j1.04°) = 2.645\underline{/1.11°}$.

ILLUSTRATIVE APPLICATION

A No. 10 A. W. G. line, whose length S = 25 miles, whose characteristic impedance at 800 cycles frequency is $Z_o = 723\underline{/-11.1°}$ vector ohms, and whose propagation constant is $p = 0.0292\underline{/71.8°}$, is short circuited at the receiving end. Calculate the short-circuit current under the assumption that the difference of potential impressed at the sending-end is V_s = 100 volts.

The expression for the desired current is

$$I = \frac{V_s}{Z_o \sinh pS}$$

For the values given above, this expression becomes

$$I = \frac{100\underline{/0°}}{723\underline{/-11.1°} \times \sinh (25 \times 0.0292\underline{/71.8°})}$$

Calculations:

$pS = 25 \times 0.0292\underline{/71.8°}$ is found as follows:

To 0.292 on D set 25 on CI.
At 71.8° on S (black) read 0.694 on D.
At 71.8° on S (red) read 0.228 on D.

Consequently

$pS = 25 \times 0.292\underline{/71.8°} = 0.228 + j0.694$.

Since 0.694 is in radians, proceed as follows:

To π on DF set 180 on CF.
At 0.694 on DF read 39.76° on CF.

$\sinh pS = \sinh (25 \times 0.0292\underline{/71.8°}) = \sinh (0.228 + j39.76°)$
is found as follows:

θ = 39.76° is less than 45°, but greater than 5.75°.
To 0.228 on scale Th set 39.76° on T (black).

Since the slide protrudes to the left, β is greater than 45° and should be read on scale T (red).

Set indicator to right index and match body and slide indices.
At indicator read 74.92° on T (red).

To x = 0.228 on scale Shl set β = 74.92° on scale S (red).
At θ = 39.76° on scale S (red) read A = 0.680.

$\sinh (0.228 + j39.76°) = 0.680\underline{/74.92°}$.

Substituting in the expression for I, we get

$$I = \frac{100\underline{/0°}}{723\underline{/-11.1°} \times 0.680\underline{/74.92°}}$$

To 723 on D set right index of C.
At 0.680 on C read 492 on D.

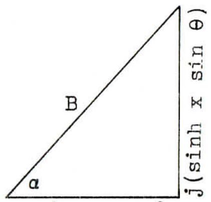
$$I = \frac{100\underline{/0°}}{492\underline{/63.82°}}$$

Set right index of C to 492 on D.
At left index of D read 0.2032 on C.

$I = 0.2032\underline{/-63.82°}$ vector amperes with reference in time phase to the impressed voltage.

(B) VECTOR EQUIVALENTS OF cosh (x + jθ).

The hyperbolic cosine of a complex number is also a complex number.



$$\cosh (x + j\theta) = (\cosh x \cos \theta) + j(\sinh x \sin \theta). \quad (9)$$

The vector value of this expression is

$$\cosh (x + j\theta) = B\underline{/\alpha}. \quad (10)$$

where $$B = \frac{\sinh x \sin \theta}{\sin \alpha} \quad (11)$$

or $$B = \sqrt{\sinh^2 x + \cos^2 \theta} \quad (12)$$

and $$\alpha = \tan^{-1}\left(\frac{\sinh x \sin \theta}{\cosh x \cos \theta}\right) = \tan^{-1} (\tanh x \tan \theta). \quad (13)$$

$$= \tan^{-1}\left(\frac{\tanh x}{\cot \theta}\right) \quad (14)$$

$$= \cot^{-1}\left(\frac{1}{\tanh x / \cot \theta}\right) \quad (15)$$

1. Slide Rule Method for Calculating the Numerical Value of cosh (x + jθ).

Equation (11) shows that the numerical value of cosh (x + jθ) can be obtained only after the angle "α" has been determined. The process consists, as seen, of a multiplication and a division by the use of the Sh, S (black) and the D scales, as follows:

To "x" on scale Sh1 or Sh2 as required, set angle "α".
(Scale ST for $\alpha < 5.75°$ or scale S (black) for $\alpha > 5.75°$).
Opposite angle "θ" on ST or S (black)
read "B" on scale D.

If in the application of the above rule, the angle "θ" is beyond scale D, exchange slide indexes and read "B" on scale D, opposite angle "θ" on scale ST or S (black).

Equation (12) shows that the value of "B" may also be considered as being the hypothenuse of a right triangle whose sides are sinh x and cos θ. For slide rule calculation the expression may be written

$$B = \sinh x + j \cos \theta \quad (12a)$$

It follows, therefore, that the value of "B" may be computed in a manner similar to that of a plane vector as explained in section I. This method has the advantage that the angle "α" of the cosh function need not be calculated first, and that it can be remembered easily. It must be kept in mind, however, that the auxiliary angle that enters into the determination of "B" by this method is not the angle of the vector "B".

Irrespective of what scheme is used to determine by slide rule method the value of "B", equation (12) shows that "B" is always larger than the largest of the two terms under the radical. Thus for cosh (0.256 + j10°) for example, the value of "B" is larger than cos 10°, while for cosh (0.256 + j80°) the value of "B" is larger than sinh 0.256 = 0.238. It follows therefore, that a comparison between the magnitudes of the two members under the radical of equation (12), will facilitate the location of the decimal point in the reading of the value of "B" on scale D.

2. Slide Rule Method for Calculating the Angle "α" of cosh (x + jθ).

Expressions (14) and (15) form the basis for the calculation of the angle "α", in a manner essentially the same as the determination of the angle of a plane vector from the value of its two right-angle components. The components in this case are tanh x and cot θ.

Since the angle "α" may have any value up to 90°, it is necessary to ascertain the conditions which determine whether it should be read on ST, T (black) or T (red). This will depend largely upon the value of cot θ, and, consequently, upon the T scale on which the angle "θ" appears. We may have therefore the following three cases:

CASE 1. $\theta < 5.75°$; $\alpha < 5.75°$.

Referring to equation (14), it is seen that since tanh x < 1 and cot θ > 10 for all values of θ < 5.75°, tan α is always smaller than 0.1. The angle "α" is therefore less than 5.75°, and should be read on scale ST.

To determine the angle "α":
To the right body index set "θ" on scale ST.
At "x" on scale Th read angle "α" on scale ST.

If "x" is beyond the slide index, it indicates that the angle $\alpha < 0.57$. In such a case:

To the left body index set "θ" on scale ST.
At "x" on scale Th, read angle "α" on scale ST.

It is important to note that in this case scale ST gives the values of the angle α between 0.057° to 0.57°. (*)

Example 11:

Evaluate cosh (0.347 + j4.5°).

To find the angle "α":

Opposite right body index set θ = 4.5° on scale ST.
At "x" = 0.347 on scale Th, read α = 1.5° on scale ST.

To find the value of "B":

To x = 0.347 on scale Shl, set α = 1.5° on scale ST.
(Since θ = 4.5° is beyond scale D, exchange slide indexes).
Opposite θ = 4.5° on scale ST, read B = 1.061 on scale D.

- -

(*) Because of certain calculations pertaining particularly to the evaluation of hyperbolic cosines of complex numbers, it is desirable that attention be called to the fact that scale D, in conjunction with scale ST also gives, with a reasonable degree of accuracy, the sines and tangents of angles smaller than 0.575°. Thus it may be used for angles between the limits of 0.0575° (3.5 minutes) and 0.57° (34 minutes), by considering the readings on this scale as decimal fractions of a degree. Thus the mark 1°, for example should be read in this case 0.1° and the mark 2.5° should be read 0.25°. It is obvious that scale CI will give in conjunction with scale ST, read as indicated above, the cotangent of such small angles, ranging in value between 1000 and 100.

The decimal place was decided upon in accordance with the suggestion stated in the preceding section, that "B" is larger than the largest number under the radical of equation (12). In this case "B" must be larger than cos 4.5°, i.e: larger than 0.997.

Hence cosh (0.347 + j4.5°) = 1.061/<u>1.5°.</u>

Example 12:

Evaluate cosh (0.21 + j1.5°).

To find the angle "α":

To the right body index set θ = 1.5° on scale ST.
(Since x = 0.21 on scale Th, is beyond the slide, exchange slide indexes).
At x = 0.21 on scale Th, read α = 0.311° on scale ST.

To obtain the value of "B":

To x = 0.21 on scale Shl, set 0.311° on scale ST.
Opposite θ = 1.5° on scale ST, read B = 1.02 on scale D.

Hence cosh (0.21 + j1.5°) = 1.02/<u>0.311°.</u>

The decimal point in the value of "B" was determined from the fact that "B" must be larger than cos 1.5° = 0.9997.

CASE 2. $5.75 < \theta < 45°$; $\alpha < 45°$.

Referring to equation (14) it is seen that for this particular case where tanh x < 1 and 10 > cot θ > 1.0, it follows that the angle "α" may have any value up to 45°. It may thus be either on scale ST or T (black), depending upon whether it is smaller or greater than 5.75°. The angle α = 5.75° when tanh x = 0.1 cot θ. Hence when tanh x < 0.1 cot θ, the angle α is less than 5.75° and should be read on scale ST; and when tanh x > 0.1 cot θ, "α" is larger than 5.75° and should be read on scale T (black).

Keeping in mind that cot θ for 5.75° < θ < 45° is on scale CI, the following rule will determine when the angle is to be read on the ST scale and when on scale T (black):

Set the left slide index to "x" on scale Th. If "θ" on scale T (black) is to the left of the right body index, "α" is less than 5.75° and should be read on scale ST. If it is to the right, then "α" is larger than 5.75° and should be read on scale T (black).

Thus for cosh (0.4 + j12°), the angle "α" is on the ST scale, while for cosh (0.4 + j25°) for example, the angle "α" is on scale T (black). The rule for evaluating the value of the angle "α" is the same as for Case 1.

Example 13:

Evaluate cosh $(0.282 + j9°)$.

To find the angle "α":
Set left slide index to 0.282 on scale Th.
Note that $\theta = 9°$ on scale T (black) is to the left of the right body index. The angle "α" is therefore on scale ST. To determine its value:
Opposite left body index set $\theta = 9°$ on scale T (black).
At $x = 0.282$ on scale Th, read $\alpha = 2.5°$ on scale ST.

To find the value of "B":
To $x = 0.282$ on scale Shl, set $\alpha = 2.5°$ on scale ST.
Opposite $\theta = 9°$ on scale S (black), read $B = 1.025$ on scale D.
The decimal point was located from the fact that "B" must be larger than cos 9°.

Hence cosh $(0.282 + j9°) = 1.025\underline{/2.5°}$.

Example 14:

Evaluate cosh $(0.178 + j16.5°)$.

To find the angle "α":
To 0.178 on scale Th, set left slide index.
The angle $\theta = 16.5°$ on T (black) is to the left of the body index, and "α" is therefore on scale ST. To obtain its value:
Opposite left body index set $\theta = 16.5°$ on scale T (black).
At 0.178 on scale Th, read $\alpha = 2.99°$ on scale ST.

To find the value of "B":
To 0.178 on scale Shl, set $\alpha = 2.99°$ on scale ST.
Note that $\theta = 16.5°$ on scale S (black), is beyond scale D.
Reset index and
Opposite $\theta = 16.5°$ on scale S (black),
read $B = 0.974$ on scale D.

Hence cosh $(0.178 + 16.5°) = 0.974\underline{/2.99°}$.

Example 15:

Evaluate cosh $(0.41 + j21°)$.

To find the value of the angle "α":
To $x = 0.41$ on scale Th, set left slide index.
Note that the angle $\theta = 21°$ on scale T black, is to the right of the body index. The angle "α" is therefore on scale T (black).
To obtain its value:
Opposite body index set $\theta = 21°$ on scale T (black).
At $x = 0.41$ on scale Th, read $\alpha = 8.48°$ on scale T (black).

To obtain the value of "B":

To $x = 0.41$ on scale Shl, set 8.48° on scale S (black).
Note that $\theta = 21°$ on scale S black is beyond scale D.
Exchange indexes, and opposite $\theta = 21°$ on scale S (black), read $B = 1.024$ on scale D.

Hence cosh $(0.41 + j21°) = 1.024\underline{/8.48°}$.

CASE 3. $\theta > 45°$; $\alpha \lesseqgtr 45°$.

Referring to equation (14), it is seen that since $\tanh x < 1.0$ and $\cot\theta < 1.0$ the angle α may be larger or smaller than 45°, and thus lie either on scale T (black) or T (red). When $\tanh x = \cot\theta$, the angle $\alpha = 45°$. Hence, when $\tanh x < \cot\theta$, the angle $\alpha < 45°$ should be read on scale T (black), and when $\tanh x > \cot\theta$, the angle $\alpha > 45°$ should be read on scale T (red.)

Since scale D from right to left gives $\cot\theta$ in conjunction with scale T (red), and also gives values of $\tanh x$ from left to right in conjunction with scale Th, we may ascertain whether "α" is larger or smaller than 45° as follows:

Match slide and body; and if "θ" on scale T (red) is then to the right of "x" on scale Th, the angle "α" should be read on scale T (black). If, on the other hand, "θ" is to the left of "x", then $\alpha > 45°$ and should be read on scale T (red). Thus for cosh $(0.4 + j65°)$, the angle $\alpha < 45°$ is on scale T (black), and for cosh $(0.4 + j75°)$, for example, the angle $\alpha > 45°$ and should be read on scale T (red).

Another way of ascertaining this is to set "θ" on scale T (red) to "x" on scale Th. Then if the slide projects to the left, the angle $\alpha < 45°$; and if the slide projects to the right, the angle $\alpha > 45°$.

To obtain the angle "α" when "θ" is to the left of "x":

Match slide and body, and set indicator over "θ" on scale T (red).
Move right slide index under indicator.
Opposite "x" on scale Th, read "α" on scale T (black).

If the angle "θ" is to the left of "x" on scale Th:
Set "θ" on scale T (red) opposite "x" on scale Th.
Opposite body index read angle "α" on scale T (red).

Example 16:

Evaluate cosh (0.185 + j56.5°).

Since "θ" is to the right of "x", the angle α < 45°.
Match slide and body, and set indicator over θ = 56.5° on scale T (red).
Move right slide index to indicator, and opposite x = 0.185 on scale Th,
read α = 15.45° on scale T (black).

To obtain the value of "B":
To x = 0.185 on scale Shl set α = 15.45° on scale S (black).
Opposite θ = 56.5° on scale S (black),
read B = 0.582 on scale D.

Hence cosh (0.185 + j56.5°) = 0.582/15.45°.

Example 17:

Evaluate cosh (0.231 + j68.6°).

Since "θ" is to the right of "x", the angle α < 45°.
Match slide and body, and set indicator over θ = 68.6° on scale T (red).
Move right slide index to indicator, and opposite x = 0.231 on scale Th,
read α = 30.1° on scale T (black).

To find the value of "B":
To x = 0.231 on scale Shl set α = 30.1° on scale S (black).
Opposite θ = 68.6° on scale S (black)
read B = 0.433 on scale D.

Hence cosh (0.231 + j68.6°) = 0.433/30.1°.

Example 18:

Evaluate cosh (0.282 + j78.5°).

Since "θ" on T (red) is to the left of "x" on scale Th, the angle α > 45°.
To x = 0.282 on scale Th, set θ = 78.5° on scale T (red).
Opposite body index read α = 53.5° on scale T (red).

To obtain the value of "B":
To x = 0.282 on scale Shl, set α = 53.5° on scale S (black).
Opposite θ = 78.5° on scale S (black)
read 0.348 on scale D.

Hence cosh (0.282 + j78.5°) = 0.348/53.5°.

Check the following: cosh (1.05 + j72°) = 1.291/67.42°.
cosh (0.25 + j36°) = 0.848/10.09°.

ILLUSTRATIVE APPLICATION

Calculate the open circuit receiving-end voltage of a No. 10 A.W.G. line, whose length S = 50 miles whose characteristic impedance at 800 cycles frequency Z_0 = 723/-11.1° vector ohms, and whose propagation constant p = 0.0292/71.8°, under the assumption that the impressed difference of potential at the sending end is V_s = 100 volts.

The expression by means of which this may be calculated is:

$$V = \frac{V_s}{\cosh pS}$$

substituting in the above expression, we get

$$V = \frac{100/0°}{\cosh (50 \times 0.0292/71.8°)}$$

cosh (50 x 0.0292/71.8°) is found as follows:

To 292 on D set 50 on CI.
At 71.8° on S (red) read 0.456 on D.
Exchange indexes:
At 71.8° on S (black) read 1.385 on D.

Hence: cosh (50 x 0.0292/71.8°) = cosh (0.456 + j1.385).

To π on DF set 180 on CF.
At 1.385 radians on DF read 79.4° on CF.

cosh (50 x 0.0292/71.8°) = cosh (0.456 + j79.4°).

θ = 79.4° is greater than 45°.

To 0.456 on Th set 79.4° on T (red).
Since the slide protrudes to the right α is greater than 45° and should be read on scale T (red).
At right index of body read α = 66.3° on T (red).
To 0.456 on Shl set 66.3° on S (black).
At 79.4° on S (black) read 0.5065 on D.

cosh (50 x 0.0292/71.8°) = 0.5065/66.3°.

Set right index of C to 0.5065 on D.
At left index of D read 197.5 on C.

Hence: V = 197.5/-66.3° vector volts lagging the sending end voltage in time phase by 66.3°.

(C) CALCULATION OF THE VECTOR VALUE OF tanh $(x + j\theta)$.

$$\text{Since } \tanh (x + j\theta) = \frac{\sinh (x + j\theta)}{\cosh (x + j\theta)} \qquad (16)$$

It follows from the discussion given in sections (A) and (B) that the hyperbolic tangent of a complex number is a vector quantity of the form

$$\tanh (x + j\theta) = D\underline{/\delta}. \qquad (17)$$

where by equation (16)

$$D\underline{/\delta} = \frac{A\underline{/\beta}}{B\underline{/\alpha}} = \frac{A}{B}\ \underline{/\beta-\alpha}. \qquad (18)$$

From the above it follows that the vector value of the tanh function of a complex number must be obtained by evaluating the sinh and the cosh functions of the same complex numbers as outlined in the preceding two sections, and taking the ratio of the vector value of the sinh to the vector value of the cosh functions.

ILLUSTRATIVE APPLICATION

Calculate the sending-end impedance of an open circuited No. 10 A.W.G. line, whose length S = 50 miles, whose characteristic impedance $Z_0 = 723\underline{/-11.1^\circ}$ and whose propagation constant is $p = 0.0292\underline{/71.8^\circ}$.

The value of the sending-end impedance is given by the expression

$$Z_s = \frac{Z_0}{\tanh pS}$$

$$\tanh pS = \tanh (50 \times 0.0292\underline{/71.8^\circ.})$$

To 0.0292 on D set 50 on CI.
At 71.8° on S (red) read 0.456 on D.
Exchange indexes:
At 71.8° on S (black) read 1.385 on D.

Hence $\qquad 50 \times 0.0292\underline{/71.8^\circ} = 0.456 + j1.385$ radians.

To π on DF set 180 on CF.
At 1.385 radians on DF read 79.4° on CF.

$$50 \times 0.0292\underline{/71.8^\circ} = 0.456 + j79.4^\circ.$$

$$\tanh (0.456 + j79.4^\circ) = \frac{\sinh (0.456 + j79.4^\circ)}{\cosh (0.456 + j79.4^\circ)}$$

$\theta = 79.4^\circ$ is greater than 45°.

To left body index set 79.4° on T (red).
At 0.456 on Th read $\beta = (90^\circ - 4.58^\circ)$ or 85.42° on scale ST.

To 0.456 on scale Shl set 85.42° or 4.58° on scale ST.
At 79.4° on S (red) read A = 1.085 on scale D.

$$\sinh (0.456 + j79.4^\circ) = 1.087\underline{/85.42^\circ}.$$

$\theta = 79.4^\circ$ is greater than 45°.

Match slide and body. $\theta = 79.4^\circ$ is to the left of x = 0.456 on scale Th. α is therefore greater than 45° and should be read on scale T (red).

To 0.456 on scale Th set $\theta = 79.4^\circ$ on scale T (red).
Opposite body index read $\alpha = 66.3^\circ$ on scale T (red).

To x = 0.456 on scale Shl set $\alpha = 66.3^\circ$ on scale S (black)
At $\theta = 79.4$ on scale S (black) read B = 0.5065 on scale D.

Hence: $\qquad \cosh (0.456 + j79.4^\circ) = 0.5065\underline{/66.3^\circ.}$

and $\qquad \tanh (0.456 + j79.4^\circ) = \dfrac{1.087\underline{/85.42^\circ}}{0.5065\underline{/66.3^\circ}}$

To 1.087 on D set 0.5065 on C.
At right index of C read 2.145 on D.

$$\tanh (0.456 + j79.4^\circ) = 2.145\underline{/19.12^\circ.}$$

Substituting in the expression for sending-end impedance, we get

$$Z_s = \frac{723\underline{/-11.1^\circ}}{2.145\underline{/19.12^\circ}}$$

To 723 on D set 2.145 on C.
At left index of C read 337 on D.

Hence: $\qquad Z_s = 337\underline{/-30.22^\circ}$

IV

INVERSE HYPERBOLIC FUNCTIONS OF COMPLEX NUMBERS

(A) SLIDE RULE CALCULATION OF $\sinh^{-1} (A\underline{/\beta})$.

To calculate the complex number $(x + j\theta)$ when the hyperbolic sine of this complex number is given, it is necessary that the vector value of the function be first split up into its components by the method outlined in section I (B). We thus get

$$\sinh (x + j\theta) = A \cos\beta + jA \sin\beta = a + jb \qquad (19)$$

where "x" and "θ" are the quantities to be calculated.

The angle "θ" of the complex function may be calculated from the expression

$$\sin \theta = \frac{\sqrt{a^2 + (1 + b)^2} - \sqrt{a^2 + (1 - b)^2}}{2} = \frac{M - N}{2} \qquad (20)$$

and the value of "x" from

$$\sinh x = \frac{a}{\cos \theta} \qquad (21)$$

To apply slide rule methods for calculating the above quantities, note that the numerator of equation (20) is the difference between the square roots of the sum of two squares. Each of the two radicals M and N may, therefore, be thought of as being the hypothenuse of a right triangle whose sides are "a" and (1 + b) for the first, and "a" and (1 - b) for the second. Their values may therefore be obtained by the methods outlined in Section I (A). It must be kept in mind, however, that these quantities are scalar quantities, and, as a consequence, the angle which enters into the calculation of these quantities is not associated with them.

The slide rule method for calculating the value of "x" is based on equation (21) and is self explanatory.

Example 19:

Given $\sinh (x + j\theta) = 1.735\underline{/34.5°}$ calculate the function.

By equation (19) we get - $\sinh (x + j\theta) = a + jb$

To 1.735 on DF set right index of slide.
At 34.5° on S (red) read 1.43 on DF.
At 34.5° on S (black) read 0.983 on DF.

$\sinh (x + j\theta) = 1.43 + j0.983.$

By equation (20) we get

$$M = \sqrt{(1.43)^2 + (1.983)^2}$$

Set slide index on 1.983 on scale D.
Move indicator to 1.43 scale D and read 54.2° on scale T (red).
With 54.2° on scale S (red) under indicator, read 2.445 opposite slide index on scale D.

Hence $M = 2.445$

In a similar manner we determine

$$N = \sqrt{(1.43)^2 + (0.017)^2} = 1.43$$

Hence by (20) we have:

$$\sin \theta = \frac{M - N}{2} = 0.5075$$

and

$$\theta = 30.5° = 0.532 \text{ radians}$$

To obtain $\sinh^{-1}\left(\frac{a}{\cos \theta}\right) = x$ by equation (21)

To a = 1.43 on scale D set θ = 30.5° on scale S (red)
At slide index read x = 1.28 on scale Sh2.

Hence $x + j\theta = 1.28 + j30.5°$

To π on DF set 180 on CF.
At 30.5 on C read 0.532 radians on D.

Hence $x + j\theta = 1.28 + j0.532$ radians.

To 1.28 on D set left index of slide.
At 0.532 on D read 22.57° on T (black).
To 0.532 on D set 22.57° on S (black).
At index of slide read 1.386 on D.

and

$\sinh^{-1} (1.735\underline{/34.5°}) = 1.386\underline{/22.57°}.$

Example 20:

Calculate $(x + j\theta)$ for $\sinh^{-1} (1.15\underline{/72.53°})$.

All calculations performed by slide rule:-

$$A\underline{/\beta} = 1.15\underline{/72.53°} = 0.345 + j1.097.$$

$$M = \sqrt{(0.345)^2 + (2.097)^2} = 2.125$$

$$N = \sqrt{(0.345)^2 + (0.097)^2} = 0.358$$

$$\sin\theta = \frac{2.125 - 0.358}{2} = 0.8835$$

$$\theta = 62.1° = 1.084 \text{ radians}$$

$$x = \sinh^{-1}\left(\frac{0.345}{\cos 62.1°}\right) = 0.683$$

Hence $x + j\theta = 0.683 + j62.1°.$

$= 0.683 + j1.084 = 1.281\underline{/57.8°.}$

or $\sinh^{-1} (1.15\underline{/72.53°}) = 1.281\underline{/57.8°.}$

(B) SLIDE RULE METHOD FOR CALCULATING $\cosh^{-1} (B\underline{/\alpha})$.

To calculate the complex number $(x + j\theta)$ when $\cosh (x + j\theta) = B\underline{/\alpha}$ is known, it is necessary to obtain first the components of $B\underline{/\alpha} = c + jd$ in the usual manner as explained in section I (B). The angle "θ" may then be calculated by the expression

$$\cos\theta = \frac{\sqrt{d^2 + (1 + c)^2} - \sqrt{d^2 + (1 - c)^2}}{2} = \frac{P - Q}{2} \qquad (22)$$

where "P" and "Q" are respectively equal to the two radicals. The value of "x" may be calculated from the equation

$$\sinh x = \frac{d}{\sin\theta} \qquad (23)$$

Equations (22) and (23) are respectively similar to (20) and (21) employed in the calculation of the inverse hyperbolic sine of a complex number. The methods of slide rule calculation of "θ" and "x" are, therefore, identical with those in the preceding case.

Example 21:

Given $\cosh (x + j\theta) = 0.582\underline{/15.42°}$, calculate $x + j\theta$.

All operations by slide rule:

$$0.582\underline{/15.42°} = 0.561 + j0.1548 = c + jd.$$

$$P = \sqrt{(0.1548)^2 + (1.561)^2} = 1.569$$

$$Q = \sqrt{(0.1548)^2 + (0.439)^2} = 0.4655$$

$$\frac{P - Q}{2} = 0.5517$$

$$\theta = \cos^{-1} 0.5517 = 56.5° = 0.986 \text{ radians}$$

By equation (23) we get

$$x = \sinh^{-1}\left(\frac{0.1548}{\sin 56.5}\right) = 0.1845$$

Hence $x + j\theta = 0.1845 + j56.5°.$

$= 0.1845 + j0.986 = 1.003\underline{/79.4°.}$

or $\cosh^{-1} (0.582\underline{/15.42°}) = 1.003\underline{/79.4°.}$

(C) SLIDE RULE CALCULATION OF $x + j\theta = \tanh^{-1} (D\underline{/\delta})$.

To calculate $x + j\theta$ corresponding to the inverse hyperbolic tangent, the vector value of $D\underline{/\delta}$ of the tanh function must be expressed in complex form:

$$D\underline{/\delta} = D\cos\delta + jD\sin\delta = m + jn. \qquad (24)$$

The value of "x" may then be calculated from the relation

$$\tanh 2x = \frac{2m}{1 + D^2} \qquad (25)$$

and the value of "θ" from

$$\tan 2\theta = \frac{2n}{1 - D^2} \qquad (26)$$

Example 22:

Given $\tanh(x + j\theta) = 0.875\underline{/24°}$ calculate $(x + j\theta)$

By (24) we have

$$0.875\underline{/24°} = 0.799 + j0.356.$$

By (25) we get

$$2x = \tanh^{-1}\left(\frac{2 \times 0.799}{1 + (0.875)^2}\right) = 1.5.$$

Hence $x = 0.75.$

By (26) we get

$$2\theta = \tan^{-1}\left(\frac{2 \times 0.356}{1 - (0.875)^2}\right) = 71.73°.$$

Hence $\theta = 35.86° = 0.626$ radians.

The function is therefore,

$$x + j\theta = 0.75 + j35.86°.$$

$$= 0.75 + j0.626 = 0.977\underline{/39.85°}.$$

or $\tanh^{-1}(0.875\underline{/24°}) = 0.977\underline{/39.85°}.$

ILLUSTRATIVE APPLICATIONS

PROBLEM 1. The series impedance Z_1 of a section of a symmetrical Π network of an artificial line is $532\underline{/64.7°}$ vector ohms at a frequency of 800 cycles per second. The Π section is known to simulate $S = 28.2$ miles of a uniform line whose characteristic impedance at the same frequency is $Z_0 = 723\underline{/-11.1°}$ vector ohms. Calculate the attenuation constant (a), the phase constant (β), and the propagation constant (p) of the line.

The relationship between the given quantities is

$$Z_1 = Z_0 \sinh(pS)$$

From the above relation we have

$$\sinh(pS) = \frac{532\underline{/64.7°}}{723\underline{/-11.1°}}$$

To 532 on D set 723 on C.
At index of C read 0.736 on D.

Consequently

$$\sinh(pS) = 0.736\underline{/75.8°}.$$

Solving by equation (19)

To 0.736 on D set right index of slide.
At 75.8° on S (black) read 0.713 on scale D.
At 75.8° on S (red) read 0.1805 on scale D.

Hence $\sinh(aS + j\beta S) = 0.1805 + j0.713$

where a = attenuation constant, and

β = phase constant.

By equation (20)

$$M = \sqrt{(0.1805)^2 + (1.713)^2}$$

$$N = \sqrt{(0.1805)^2 + (0.287)^2}$$

To 1.713 on scale D, set left index of slide.
At 0.1805 on D read 83.985° on scale T (red).
To 0.1805 on D, set 83.985° on S (red).
At left index of slide read M = 1.723 on D.

To 0.287 on D set right index of slide.
At 0.1805 on D read 57.81° on T (red).
To 0.1805 on D, set 57.81° on S (red).
At right index of slide read N = 0.339 on D.

$$\frac{M - N}{2} = \frac{1.723 - 0.339}{2} = 0.692$$

$$\theta = \beta S = \sin^{-1} 0.692 = 43.8°$$

Set indicator to 0.692 on C.
Under indicator read 43.8° on S (black).

To π on DF set 180 on CF.
At 43.8 on CF read 0.764 radians on DF.

$$\beta S = 0.764 \text{ radians}$$

$$\beta = \frac{0.764}{28.2}$$

To 0.764 on D set 28.2 on C.
At left index of slide read 0.0271 on D.

The phase constant is -

$$\beta = 0.0271 \text{ radians per mile.}$$

By equation (21) we similarly get

$$x = aS = \sinh^{-1}\left(\frac{0.1805}{\cos 43.8°}\right)$$

To 0.1805 on D set 43.8° on S (red).
At right index of slide read 0.2477 on scale Shl.

$$x = 0.2477$$

The attenuation constant is therefore

$$a = \frac{0.2477}{28.2} = 0.00878 \text{ hyperbolic radians per mile.}$$

The propagation constant is

$$p = a + j\beta = 0.00878 + j0.0271.$$

To 0.0271 on D set left index of slide.
At 0.00878 on D read 72.05° on T (red).
To 0.00878 on D, set 72.05° on S (red).
At index of slide read 0.0285 on D.

$$p = 0.0285\underline{/72.05°}.$$

PROBLEM 2. The impedance Z_1 of the series branch of a symmetrical T structure is $450\underline{/50°}$ vector ohms and the impedance of the shunt branch is $Z_2 = 529\underline{/-90°}$ vector ohms at 1000 cycles per second frequency. Calculate the propagation constant (p) of the T network.

The relation between the given quantities and the propagation constant p of the circuit is

$$\cosh p = 1 + \frac{Z_1}{Z_2}$$

Hence

$$\cosh p = 1 + \frac{450\underline{/50°}}{529\underline{/-90°}}$$

To 450 on scale D set 529 on scale C.
At index of slide read 0.851 on D.

$$\cosh p = 1 + 0.851\underline{/140°}.$$
$$= 1 - 0.851\underline{/-40°}.$$

To 0.851 on D set right index of slide.
At 40° on S (black) read 0.546.
At 40° on S (red) read 0.651.

$$1 - 0.651 = 0.349$$

$$\cosh p = 0.349 + j0.546$$

By equation (22) we have

$$P = \sqrt{(0.546)^2 + (1.349)^2} = 1.455$$
$$Q = \sqrt{(0.546)^2 + (0.651)^2} = 0.850$$

$$\frac{P - Q}{2} = 0.3025$$

By equation (22) we also get

$$\theta = \cos^{-1} \ (0.3025)$$

To 0.3025 on C set indicator.
At indicator read 72.4° on S (red).
To π on DF set 180 on CF.
At 72.4 on CF read 1.264 radians on DF.

$$\theta = 1.264 \text{ radians}$$

By equation (23) we have

$$x = \sinh^{-1}\left(\frac{0.546}{\sin 72.4°}\right)$$

To 0.546 on D set 72.4° on S (black).
At right index of slide read 0.5455 on scale Shl.

Hence, the propagation constant is

$$p = x + j\theta = 0.5455 + j1.264.$$

To 1.264 on D set left index of slide.
At 0.5455 on D read 66.65° on T (red).
Move 66.65° on S (red) to indicator.
At left index of slide read 1.376 on D.

Hence $$p = 1.376\underline{/66.65°}.$$

PROBLEM 3. The measured value of the sending-end impedance Z_s of a symmetrical T circuit at 1000 cycles per second frequency is $1678\underline{/-70.8°}$ vector ohms. The image impedance Z_o of the same circuit is $723\underline{/-11.1°}$ vector ohms. Calculate the propagation constant (p) of the circuit.

The relation between the given quantities and the propagation constant is

$$Z_s = \frac{Z_o}{\tanh p}$$

For the given data we have

$$\tanh p = \frac{Z_o}{Z_s} = \frac{723\underline{/-11.1°}}{1678\underline{/-70.8°}}$$

To 723 on D set 1678 on C.
At left index of C read 0.431 on D.

$$\tanh p = 0.431\underline{/59.7°}.$$

To 0.431 on D set right index of slide.
At 59.7° on S (black) read 0.372 on D.
At 59.7° on S (red) read 0.2175 on D.

$$\tanh p = 0.2175 + j0.372$$

By equation (25) we get

$$x = \frac{1}{2}\tanh^{-1}\left(\frac{2 \times 0.2175}{1 + (0.431)^2}\right)$$

Indicator to 0.431 on D.
At indicator read 0.186 on A.
To 2 on D set 1.186 on C.
At 0.2175 on C read $2x = 0.385$ on scale Th.

$$x = 0.1925$$

Similarly by equation (26) we get

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2 \times 0.372}{1 - (0.431)^2}\right)$$

To 2 on DF set 0.814 on CF.
Move indicator to 0.372 on C.
Match slide and body.
At indicator read $2\theta = 42.4°$ on T (black).

$$\theta = 21.2° = 0.370 \text{ radians}$$

Hence, the propagation constant of the circuit is

$$p = x + j\theta.$$

$$= 0.1925 + j0.370.$$

To 0.370 on D set right index of slide.
At 0.1925 on D read 62.5° on T (red).
To 0.1925 on D set 62.5° on S (red).
At right index of slide read 0.417 on D.

$$p = 0.417\underline{/62.5°}.$$