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A Slide Rule for Vector Calculations

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ELECTRICAL networks such as filters, attenuation equalizers, transformers, balancing networks and speech delay circuits are used in large numbers in many parts of the telephone plant. For example, filters separate the voice and carrier frequency currents at the ends of each toll circuit on which carrier currents are superposed, and attenuation equalizers are used on broadcasting circuits to give sufficiently uniform transmission over the wide range of frequencies necessary for high grade radio programs. Transformers are extensively used at junction points in circuits where the impedances are unequal.

These networks are made up of resistances, inductances and capacities and in some cases the number of elements reaches a hundred or more. In order to ascertain their performance accurately, a large amount of mathematical computation is involved. Any considerations simplifying the work of computation are therefore deserving of serious thought, particularly when a considerable saving in time is thereby effected.

In most cases the solution of electrical networks involving alternating voltages can be simplified by expressing the voltages and currents as vectors, to show their magnitudes and the phase relationships between them. The voltages and currents are represented in polar form, with magnitude and direction given, or in rectangular form, by complex numbers giving the real and imaginary components. Partly offsetting the advantages of this notation, however, is the work frequently involved in solving the equations, particularly the routine operations in transferring them between the polar and rectangular forms.

If a complex quantity is expressed in polar form, its length r and the angle θ which it makes with the hori-

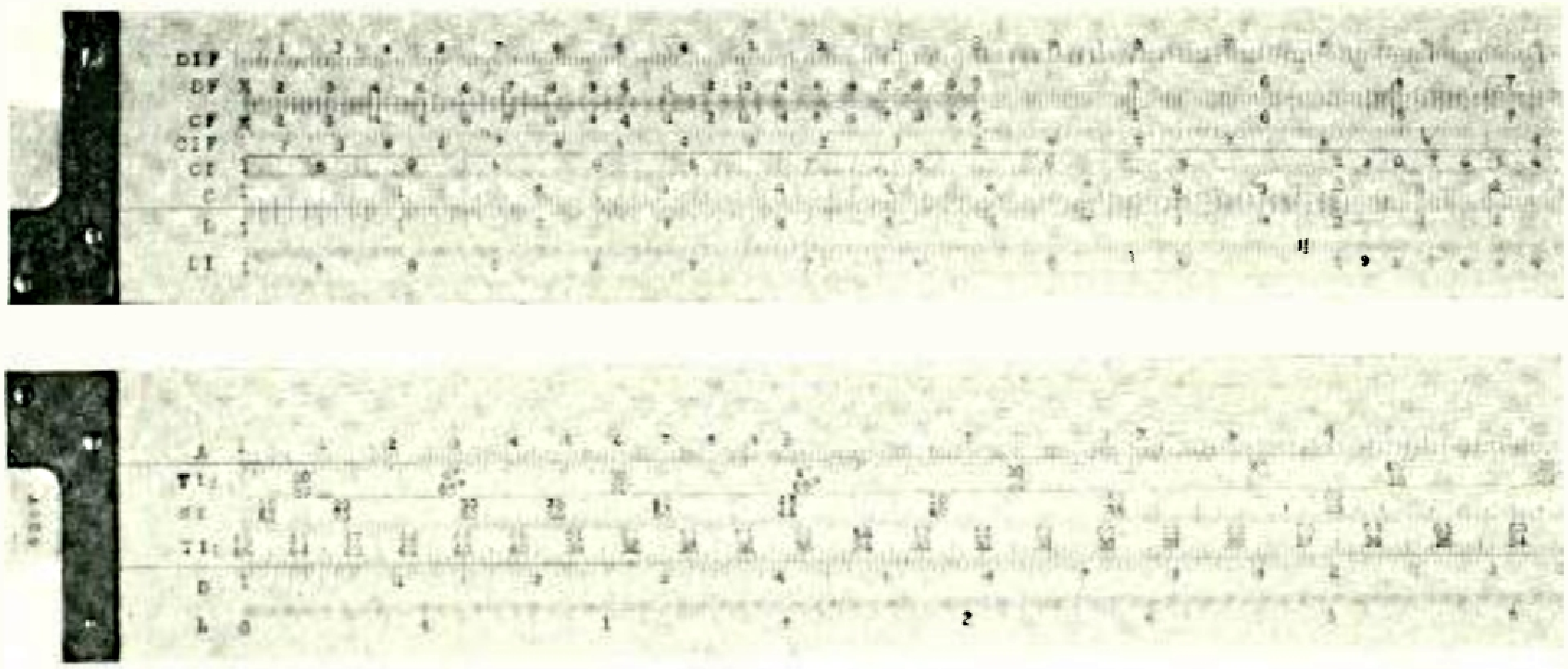

Fig. 1—Above, the front of the new rule, with the added scales, DIF and DI, on the upper and lower stationary sections; below, reverse side, showing the inverted trigonometric scales on the slide

zontal axis are given, usually in the notation $r \underline{|\theta}$. In the rectangular form the number is expressed as $a + jb$. The relationship between these notations, shown in Figure 2, is such that either can be changed to the other by the simplest trigonometric equations. Simple as it is, however, the change involves a number of separate computations, and when there is an extensive series of equations to be solved the work of computation is long and tedious.

Various mechanical devices, some of them specially designed for the purpose, have been used for reducing the time taken by this operation. An ordinary duplex slide rule is used most commonly, but it was designed for general use rather than for this particular operation, and numerous settings of the rule are required. An instance is that the tangent scale between the values 0.1 and 1.0 runs the full length of the slide, while in the same linear distance the sine scale extends between the values 0.01 and 1.0. Since both sine and tangent of the angle are used in each transformation, the time lost in resetting the rule on account of this discrepancy between the two scales is considerable. Other similar situations require more settings of the rule than are inherently necessary for these transformations.

Accordingly in 1916 F. A. Hubbard* designed a special slide rule for vector transformations, with scales provided on it which reduced to three the number of settings of slide and hair line for a complete transformation. Such a rule was later made, and it proved a valuable time saver. It had however two disadvantages. Although its length was twenty inches, the scales were so arranged that the accuracy was only that of a ten-inch rule. Also it was necessary to set up the numerical factors on an inverted scale, reading from right to left; that increased the chance of error from carelessness on the part of users, since the reverse side of the rule was also used for ordinary operations of multiplication and division.

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Another special slide rule, intended particularly for solving equations involving hyperbolic functions, was designed by Professor M. P. Weinbach of the University of Missouri, and was described in the Journal of the A. I. E. E. for May, 1928. Problems involving ordinary complex numbers may be solved as well, but four settings are required for a change from rectangular to polar form, and six settings are needed for the reverse transformation.

To expedite our filter computations, a twenty-inch rule was designed with the scales arranged especially for vector transformations. Preliminary arrangements for its manufacture were made with the Keuffel and Esser Company early in 1928, and in the summer six of the rules were made for the Research Department and the Apparatus Development Department. In this rule only one setting of the slide and two settings of the hair line are needed in changing a complex number from either form to the other. The numbers are set on an ordinary logarithmic or D scale, and the operations are performed in the same manner as are multiplication or division on an ordinary slide rule. The results are obtained on the D scale—the full-length logarithmic scale on the stationary part of the rule—and since the rule is of duplex type, additional operations such as multiplication or division may when desired be carried on without a resetting of the complex number.

The front of the rule, shown in Figure 1, has the scales usual for polyphase duplex rules, and has in addition the reciprocal or inverted scales DIF and DI on the stationary part of the rule. These scales, giving the reciprocal of a result directly, end the need for a separate operation to secure a reciprocal. They are specially useful in such operations as finding capacitive reactance and admittance. For obtaining the reactances in a network containing both inductance and capacity only a single setting of the slide is made, and then all the reactances are obtained by moving the hair line.

It is on the reverse side of the slide that the principal changes have been made. Trigonometric scales for sine and tangent have both been inverted, and are both referred to the DI scale—the inverted D scale—rather than to the A and D scales, respectively, as before. Referring both to the same scale does away with the

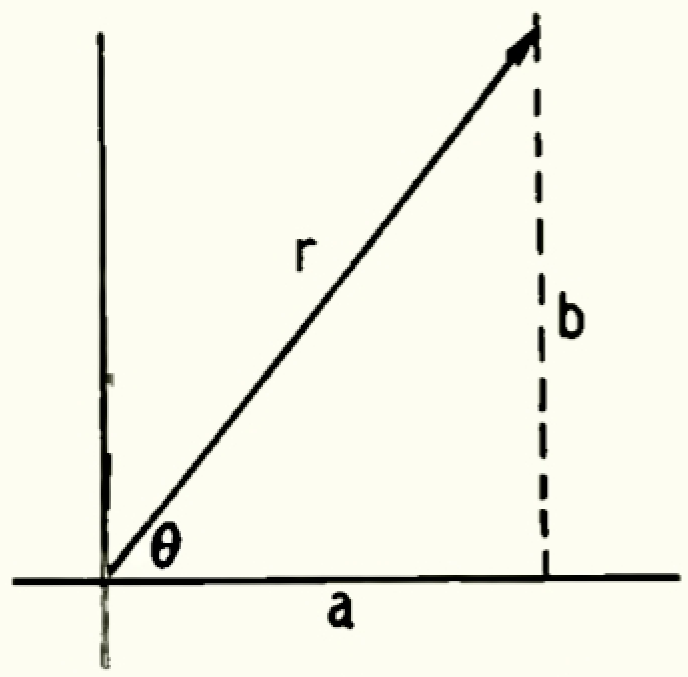

Fig. 2—In a complex number, r *is the vector and* θ *the angle it makes with the axis of real numbers;* a *is the real component and* jb *the imaginary component. With either* a *and* b *or* r *and* θ *known, conversion between polar and rectangular form consists in finding the other pair of values*

need for resetting the rule to use one function when the other has just been ascertained. The principal tangent scale TI_1, runs from 5°43′, the angle whose tangent is 0.1, to 45°, just as in all polyphase rules, and in addition an auxiliary tangent scale TI_2 running from approximately 34′ to 5° 43′ is given, to cover the angles whose tangents are between 0.01 and 0.1.

Fig. 3—Beginning of a conversion from rectangular to polar coordinates. When the slide is set on 1195 *and the hair line on* 1530 *on scale D, angle θ is given as* 52° *on scale TI_1. The later example is also illustrated here. With the slide set on* 119.5 *and the hair line on* 1530, *the angle is read on scale TI_2—*85° 32′

The sine scale extends between the limits 0.1 and 1.0 only—that is, between angles 5° 45′ and 90°. For angles less than 5° 45′ however sine and tangent are so close together that scale TI_2 may be used in place of an auxiliary sine scale without serious error.

Convenience of this rule in changing a complex number from rectangular to polar form can best be shown by a numerical example, such as 1195 +j1530. First the angle θ is obtained through its tangent. The index of the slide is set on the smaller number, 1195, on the D scale, and the hair line is moved to the larger number, 1530, on the same scale. Then the point on the slide beneath the hair line marks the ratio of the two sides, which is the tangent or cotangent of the angle, depending on which side was larger; in this case the ratio is the tangent. Then the point on the tangent scale beneath the hair line is read directly as the angle wanted—in this case, 52°. With the angle known, the length of the vector is next obtained by dividing the length of the vertical component by the sine of the angle, or the horizontal component by the cosine of the angle. In this case the horizontal side, 1195, is already set on the rule; accordingly the hair line is moved to 52° on the divisions of the sine scale SI corresponding to cosine values, and the length of the vector, 1942, is then found under the hair line on scale D.

In the example given, the ratio of the larger to the smaller component was less than 10, and the tangent scale TI_1 was therefore used. When the ratio is between 10 and 100 the

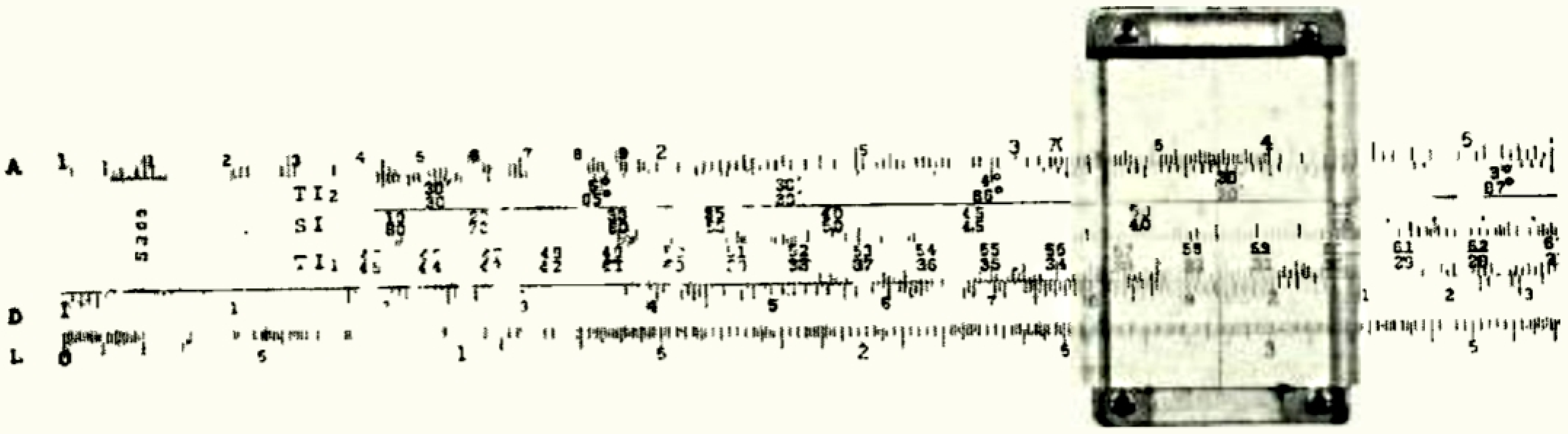

Fig. 4—With θ learned, the slide is left at 1195 *and the hair line moved to* 52° *on scale SI. Then the vector* r *is given on scale D; its value is* 1942

procedure is the same except that the other tangent scale on the slide, TI_2, is used. It gives the tangents of angles between 85°43′ and 89°26′, and of course the cotangents of the complementary angles. A case in which this scale would be used is the complex number 119.5 + j1530. The index of the slide is set on stationary scale D at the smaller number, 119.5, just as before and the hair line moved on the same scale to the larger number, 1530. Then the ratio of the two sides appears on the slide, and on scale TI_2 it represents the tangent of angle θ — in this case, 85°32′.

The length of the vector cannot be obtained directly on the rule, since the sine scale has not been continued between 0.01 and 0.1 on a scale comparable to TI_1. Should the vector be wanted closely it can be obtained readily by auxiliary computation, but ordinarily it is sufficiently accurate to take the larger component as the value of the vector. At most the error is one-half of one per cent, when the ratio of the components is just ten, and as the ratio increases the error drops rapidly, so that when the larger side is about twenty-two times the smaller, the error is only a tenth of one per cent. The factor by which the larger component is to be increased is shown in Figure 5; when the desired accuracy justifies the extra computation, the larger component is increased by the fractional per cent through additional settings of the rule or by mental calculation. In the example shown, the increase is 0.3%, and length of the vector 1535.

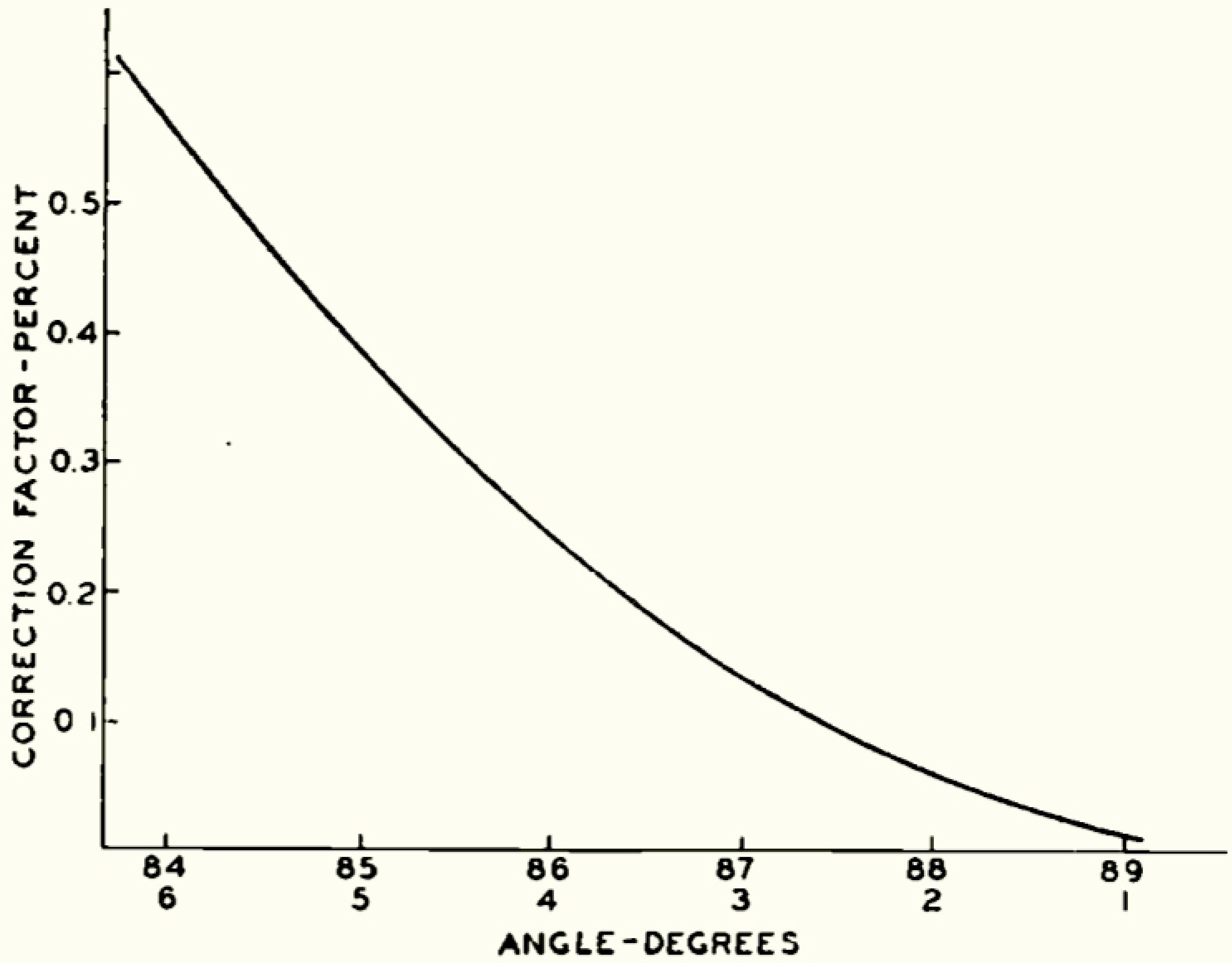

Fig. 5—For values of θ not given on scale SI, the larger component is increased in accordance with this curve to get the vector

When a complex number in polar form, as $1942\ \underline{|52^\circ}$, is to be changed to rectangular form, the operations are the reverse of those already described. The hair line is set on the length of the vector, 1942, on scale D; the angle 52°, on sine scale SI is drawn beneath the hair line; then the hair line is moved to 52° on tangent scale TI_1. Thereupon the two components are read directly on scale D —one of them beneath the index of the slide, and the other at the hair line. That at the index, 1195, is the vector multiplied by the cosine of angle θ and is accordingly the real component; the number at the hair line, 1530, is the imaginary component—the vector multiplied by the cosine of the angle and then multiplied by the tangent.

Although the rule was originally

intended primarily for filter computations, its usefulness is not restricted to those engaged in designing filters. In ten-inch size it would be in many ways advantageous for general engineering and student use. For such a field however, certain modifications would be desirable. On the tangent and sine scales the angles should be divided decimally rather than in minutes and seconds, to facilitate interpolation and to simplify addition and subtraction of angles. The scale of equal divisions, or L scale, might well be placed on the front of the rule, and the space on the reverse side vacated by this change used for exponential, or log-log, scales. With that arrangement there would be no need for a cube or K scale, and little if any need for a square — A — scale. Another change, though one whose general advisability is somewhat open to question, is a further modification of the sine scale. That part of the scale between 0.707 and 1.0, or between 45° and 90°, is never used in transforming complex numbers, and its only value is to give directly the numerical values of sines and cosines within its range. Since these could easily be found from other parts of the sine scale by additional operations, this part of the SI scale could be replaced by an SI_2 scale extending from 5° 45′ down to 4°8′. If that were done, the angle whose vector could be obtained directly on the rule would be increased from that with the tangent 10 to that with the tangent 14, and the greatest error which would be present if the vector were not corrected in accordance with the curve of Figure 5 would be 0.3% rather than 0.5%.

Since complex numbers enter into computations in many other fields than those of telephone filters, a rule so made would speed up computations of many sorts, and at the same time fit the general types of computation for which a slide rule is commonly used.

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—*Barron's, May* 13, 1929.